Snow College Mathematics Contest

$\boxed{\text{key}}$

April 2, 2013 **Senior Division: Grades 10-12** **Form:** $\boxed{\text{T}}$

Bubble in the single best choice for each question you choose to answer.

1. When rolling two fair dice, what is the probability of getting a sum of 7?

(A) $\frac{1}{7}$

(B) $\frac{1}{36}$

$\boxed{\text{(C)}}$ $\frac{1}{6}$

(D) $\frac{7}{36}$

(E) $\frac{6}{7}$

$\boxed{\text{SOLN}}$ There are 36 total possible outcomes from rolling two fair dice; six of them have a sum of 7.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

□

2. Find the sum: $\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\ldots$

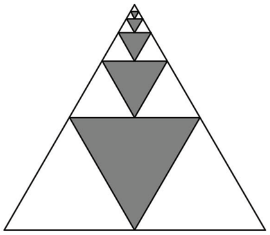
(A) $\frac{4}{3}$

$\boxed{\text{(B)}}$ $\frac{1}{3}$

(C) $\frac{5}{4}$

(D) $\frac{1}{2}$

(E) ∞



$\boxed{\text{SOLN}}$ The sum of $1+r+r^2+r^3+r^4+\ldots$ is $\frac{1}{1-r}$ for $|r|<1$. In this case $r=\frac{1}{4}$ so the sum is $\frac{1}{1-(1/4)}=\frac{1}{3/4}=4/3$. Our case doesn't have the leading 1, so the final answer is $4/3-1=1/3$.

Or, let $S=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$. Then $4S=1+S \Rightarrow S=1/3$.

Neat proof by picture: each shaded triangle is 1/4 the area of the larger one; yet each shaded triangle is 1/3 of the area of a horizontal stripe. □

3. The Pauli spin matrices σ_1, σ_2, and σ_3 appear in quantum mechanics. They are

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \sigma_2=\begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix} \quad \sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

The *trace* of a matrix is the sum of the elements on the main diagonal: $\mathrm{Tr}(A)=\Sigma_{i=1}^{i=n} a_{ii}$. What is $\mathrm{Tr}(\sigma_1+\sigma_2+\sigma_3)$?

$\boxed{\text{(A)}}$ 0

(B) -1

(C) i

(D) $-$i

(E) I

$\boxed{\text{SOLN}}$
$$\sigma_1+\sigma_2+\sigma_3=\begin{bmatrix}1 & 1-\mathrm{i}\\ 1+\mathrm{i} & -1\end{bmatrix}$$

$1+(-1)=0$ Note: $\mathrm{Tr}(\sigma_1+\sigma_2+\sigma_3)=\mathrm{Tr}(\sigma_1)+\mathrm{Tr}(\sigma_2)+\mathrm{Tr}(\sigma_3)$ □

4. Referring to the figure, find AC.

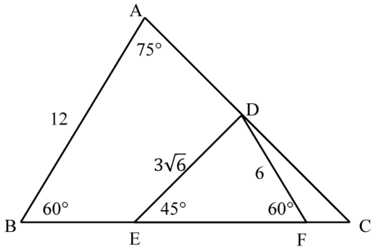
(A) $4\sqrt{6}$

$\boxed{\text{(B)}}$ $6\sqrt{6}$

(C) $8\sqrt{3}$

(D) $8\sqrt{6}$

(E) $12\sqrt{2}$



$\boxed{\text{SOLN}}$ Geometry Soln: The sum of the angles in a $\triangle$ is 180°, so $m(\angle C)=45°$. $\triangle ABC$ is similar to $\triangle DFE$ by angle-angle. $\therefore$ AC/DE = AB/DF, so $\mathrm{AC}/3\sqrt{6}=12/6 \implies \mathrm{AC}=6\sqrt{6}$. Trig Soln: Law of sines: $\frac{12}{\sin 45°}=\frac{\mathrm{AC}}{\sin 60°}$

□

5. Any point on the earth makes one revolution about the earth's axis in 24 h. If the radius of the earth is 3950 mi, what is the linear velocity, in miles per hour, of a point on the equator? Use approximations.

(A) 42 mph

(B) 165 mph

(C) 392 mph

(D) 518 mph

$\boxed{\text{(E)}}$ 1034 mph

$\boxed{\text{SOLN}}$ Linear velocity is the rate at which distance is changing. In 24 hours a point on the equator has traveled the circumference of the earth.

$$v = \frac{2\pi r}{\Delta t} = \frac{2\pi(3950\,\text{mi})}{24\,\text{h}} = 1034\,\text{mph}$$

$\frac{2\pi}{24} \approx \frac{1}{4}$; $3950 \approx 4000$ $v \approx 1000$ $\square$

6. Some students rented a boat during spring break. They went 60 mph from the dock to an island 60 miles away. They immediately turned around and returned at a more leisurely 30 mph. What was their average speed for the whole trip?

(A) 50 mph

(B) 48 mph

(C) 45 mph

(D) 42 mph

$\boxed{\text{(E)}}$ 40 mph

$\boxed{\text{SOLN}}$ speed $= \frac{\text{dist}}{\text{time}} = \frac{120\,\text{mi}}{3\,\text{h}} = 40\,\text{mph}$ $\square$

7. What is the output of the following BASIC computer program with some natural number n given as input?

```
10 input n
20 f = 1
30 for i = 1 to n
40 f = f * i
50 next i
60 print f
```

$\boxed{\text{(A)}}$ $n!$

(B) the prime factors of n

(C) least common multiple of n and f

(D) nth triangular number

(E) n^n

$\boxed{\text{SOLN}}$ This is an iterative computation of n factorial. (If f were initialized to 0 and the operation in line 40 were $+$ instead of $*$ then D would be the right answer.) $\square$

8. What is the remainder when $x^3 - 2x^2 + 4$ is divided by $x + 2$?

$\boxed{\text{(A)}}$ -12

(B) 0

(C) 4

(D) 6

(E) 12

$\boxed{\text{SOLN}}$ Of course, one could perform long division to get the answer. Synthetic division is a quicker method. However, the remainder theorem says that the remainder upon division (by a linear factor) is equal to the functional value at that point. The linear factor $x+2$ corresponds to the point $x = -2$ and $f(-2) = -8 - 8 + 4 = -12$. $\therefore r = -12$. $\square$

9. Powers of two are additive building blocks of the whole numbers; that is, each whole number can be expressed as the sum of powers of two (with all different powers) in a unique way. For example, $10 = 2^3 + 2^1$. What is the sum of the exponents in such an expression for 127?

   (A) 11

   (B) 13

   (C) 15

   (D) 18

   $\boxed{\text{(E)}}$ $21 = 6+5+4+3+2+1+0$

   $\boxed{\text{SOLN}}$ $2^6+2^5+2^4+2^3+2^2+2^1+2^0 = 127 = 64+32+16+8+4+2+1$ □

10. Two black balls and one white ball (identical except for color) are in a bag. A ball is drawn, its color recorded, and then the ball is replaced in the bag and the balls are mixed up. This process is done four times. What is the probability that of the four balls drawn that exactly two of them are black?

    (A) $\frac{4}{81}$

    (B) $\frac{1}{2}$

    $\boxed{\text{(C)}}$ $\frac{8}{27} = 6(\frac{2}{3})^2(\frac{1}{3})^2$

    (D) $\frac{8}{81}$

    (E) $\frac{3}{8}$

    $\boxed{\text{SOLN}}$ A probability tree diagram will eventually yield the answer. The probability will be ${}_nC_r\, p^r q^{n-r}$ with $n = 4$, $r = 2$, $p = P(\text{black}) = 2/3$, $q = P(\text{white}) = 1/3$. ${}_4C_2 = 6$ □

11. Which is a solution of the equation?

    $$\cos^2\theta = \frac{2+\sqrt{3}}{4}$$

    (A) $\frac{\pi}{18}$

    $\boxed{\text{(B)}}$ $\frac{\pi}{12}$

    (C) $\frac{\pi}{9}$

    (D) $\frac{\pi}{6}$

    (E) $\frac{\pi}{3}$

    $\boxed{\text{SOLN}}$ Use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ on the left side; then reduce to $\cos 2\theta = \frac{\sqrt{3}}{2}$. □

12. Characterize the roots of $x^3+6x^2+11x+6$.

    (A) no negative roots

    $\boxed{\text{(B)}}$ no positive roots

    (C) no real roots

    (D) 1 positive root, 2 negative roots

    (E) 1 negative root, 2 positive roots

    $\boxed{\text{SOLN}}$ Descartes' rule: no variation in sign for $P(x) \Longrightarrow$ no positive roots. $P(-x)$ has 3 changes in sign $\Longrightarrow$ 1 or 3 negative roots (there are 3). □

13. If 2 is a solution of $x^3 + hx + 10 = 0$, then what is h?

    (A) 10

    (B) 9

    (C) 2

    (D) -2

    $\boxed{\text{(E)}}$ -9

    $\boxed{\text{SOLN}}$ $2^3 + 2h + 10 = 0 \Rightarrow 2h = -18$ □

14. In the figure $\overline{AB} = \overline{AC}$, $\angle BAD = 30°$, and $\overline{AE} = \overline{AD}$. Find the measure of angle x.

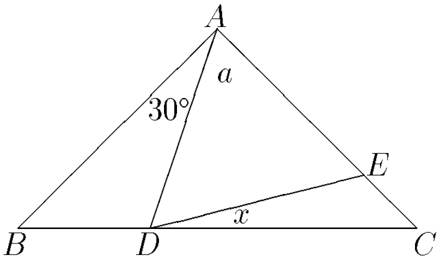
(A) $7\frac{1}{2}°$

(B) $10°$

(C) $12\frac{1}{2}°$

$\boxed{\text{(D)}}$ $15°$

(E) $20°$



$\boxed{\text{SOLN}}$ Call $\angle DAE = a$.
$\angle BCA = \frac{1}{2}(180 - 30 - a) = 75 - \frac{a}{2}$
$\angle ADE = \frac{1}{2}(180 - a) = 90 - \frac{a}{2}$
$a + \angle ADE + x + \angle BCA = 180$
$a + (90 - \frac{a}{2}) + x + (75 - \frac{a}{2}) = 180$ □

15. If a recipe calls for $2\frac{3}{4}$ c flour to make 3 dozen cookies, how much flour is required to make 7 dozen cookies?

(A) $4\frac{7}{12}$ c

(B) $5\frac{1}{2}$ c

$\boxed{\text{(C)}}$ $6\frac{5}{12}$ c

(D) $7\frac{7}{4}$ c

(E) $19\frac{1}{4}$ c

$\boxed{\text{SOLN}}$ Use ratios: $\frac{2\frac{3}{4}\text{ c}}{3\text{doz.}} = \frac{x\text{ c}}{7\text{doz.}}$ □

16. Find the solution set.

$$\log_5(x+6) + \log_5(x-6) = 3$$

$\boxed{\text{(A)}}$ $\{\sqrt{161}\}$

(B) $\{\frac{125}{3}\}$

(C) $\{\frac{183}{5}\}$

(D) $\{161\}$

(E) $\{-\sqrt{161}, \sqrt{161}\}$

$\boxed{\text{SOLN}}$ $\log_5(x+6)(x-6) = 3$
$(x+6)(x-6) = 5^3$
$x^2 - 36 = 125 \implies x = \pm\sqrt{161}$, but $-\sqrt{161}$ is not in the domain. □

17. What is the area between $f(x) = \sin x + 1$ and $g(x) = \cos x + 1$ between any two successive crossings?

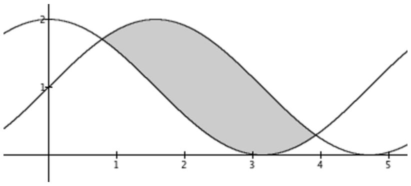
(A) π

(B) $\pi/2$

(C) $3\pi/2$

(D) $2\pi/3$

$\boxed{\text{(E)}}$ $2\sqrt{2}$



$\boxed{\text{SOLN}}$ The crossings shown are at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, which are the limits of integration.

$$\int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx =$$

$$\Big[-\cos x - \sin x \Big]_{\pi/4}^{5\pi/4} =$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

□

18. a modulo b means the remainder when a is divided by b; for example, 14 modulo 3 = 2. The set $\mathbb{F} = \{0, 2, 4, 6, 8\}$ together with the binary operations of addition and multiplication modulo 10 is a *field*. Which is the unity element (multiplicative identity)?

(A) 0

(B) 2

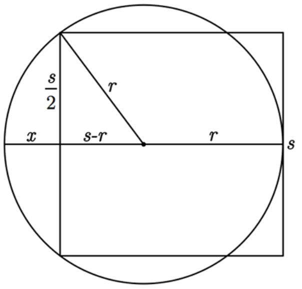
(C) 4

$\boxed{\text{(D)}}$ 6

(E) 8

$\boxed{\text{SOLN}}$ For any element $x \in \mathbb{F}$,
$6 \cdot x = x \cdot 6 = x \mod 10$ □

19. A square of side 8 meets a circle tangent to a side with opposite corners incident with the circle. Find x along the circle's diameter.



$\boxed{\text{(A)}}$ 2

(B) $2\frac{1}{8}$

(C) $2\frac{3}{8}$

(D) $2\frac{1}{2}$

(E) $2\frac{5}{8}$

$\boxed{\text{SOLN}}$ Use the Pythagorean theorem.

$$\left(\tfrac{s}{2}\right)^2 + (s-r)^2 = r^2$$

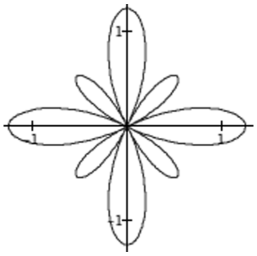
$$\tfrac{s^2}{4} + s^2 - 2sr = 0$$

$$\tfrac{5}{8}s = r$$

$$x = r - (s-r) = 2r - s = \tfrac{1}{4}s$$

□

20. Which polar equation best represents the graph for $0 \le \theta \le 2\pi$?



(A) $r = 8\theta$

(B) $r = \theta^2$

(C) $r = \sin 4\theta$

$\boxed{\text{(D)}}$ $r = \cos 4\theta + \frac{1}{4}$

(E) $r = \sin 8\theta$

$\boxed{\text{SOLN}}$ $r = \cos 4\theta$ produces 8 equal-sized lobes. Adding the $\frac{1}{4}$ makes the lobes bigger when $\cos 4\theta > 0$ and smaller when $\cos 4\theta < 0$. □

21. In logic we use $\wedge$ for "and," $\vee$ for "or," and $\neg$ for "not." Which of the following is logically equivalent to $(P \vee Q) \wedge (Q \vee R)$?

(A) $P \wedge R$

$\boxed{\text{(B)}}$ $(P \wedge R) \vee Q$

(C) $(P \wedge Q) \vee (Q \wedge R)$

(D) $(\neg Q) \wedge (P \vee R)$

(E) $P \wedge Q$

$\boxed{\text{SOLN}}$ Two statements are equivalent if they have the same truth table. □

22. An operation on a row of seven circles, where each circle is either black or white, consists of choosing any **two** of the circles and changing the colors of each of them (i.e., from black to white, or from white to black).

○ ● ○ ○ ○ ● ●

Which of the following rows of circles <u>cannot</u> be obtained by any repeated application of such operations upon the row above?

(A) ○ ● ○ ○ ● ○ ●

(B) ● ○ ● ● ● ● ○

(C) ● ● ● ● ● ● ●

(D) ○ ○ ○ ○ ● ● ●

$\boxed{\text{(E)}}$ ● ○ ● ○ ● ○ ●

$\boxed{\text{SOLN}}$ Any move preserves the parity of the number of black circles. That is, since the number of black circles starts as odd, it must remain odd. □

23. The sky is blue because molecules in the atmosphere scatter light. The intensity of scattered light I is inversely proportional to the fourth power of the wavelength λ.

$$I \propto \frac{1}{\lambda^4}$$

Violet light ($\lambda = 400\,\text{nm}$) is scattered how much more than redder-than-red infrared light ($\lambda = 800\,\text{nm}$)?

(A) twice as much

(B) four times as much

(C) eight times as much

$\boxed{\text{(D)}}$ sixteen times as much

(E) thirty-two times as much

$\boxed{\text{SOLN}}$ $$\frac{I_{\text{violet}}}{I_{\text{infrared}}} = \left(\frac{\lambda_{\text{infrared}}}{\lambda_{\text{violet}}}\right)^4 = 2^4$$

□

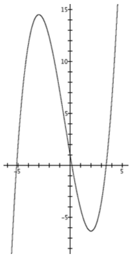
24. In the grid each cell contains one of the digits 1 to 5 so that each row and each column has exactly one of each digit. Find the entry in row 3, column 4.

$\boxed{\text{(A)}}$ 1
(B) 2
(C) 3
(D) 4
(E) 5

1	2	5	4	4
2	5	4	3	1
5	4	3	**1**	2
4	3	1	2	5
3	1	2	5	4

$\boxed{\text{SOLN}}$ Sudoku! □

25. For what value of x is there a local minimum of $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 1$?



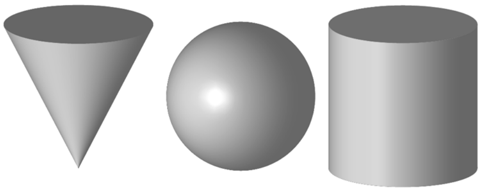
(A) $\frac{11}{3}$
(B) $\frac{8}{5}$
(C) $-\frac{5}{8}$
$\boxed{\text{(D)}}$ 2
(E) -3

$\boxed{\text{SOLN}}$ Local minimum means the function increases in both directions. To locate it, take the derivative of the function, factor, and set equal to zero.

$$y' = x^2 + x - 6 = (x-2)(x+3) = 0$$

Thus, the derivative is 0 at $x = -3, 2$. The second derivative, $y'' = 2x + 1$, is positive at $x = 2$. □

Questions 26–27: The cone, ball, and cup all have the same height, and for each one the height and diameter are equal.



$V_{\text{cone}} = (\frac{1}{3})(\pi R^2)(2R) = \frac{2}{3}\pi R^3$
$V_{\text{sphere}} = \frac{4}{3}\pi R^3$
$V_{\text{cup}} = (\pi R^2)(2R) = 2\pi R^3$

26. How many cones-full of water will fill the cup?

(A) 1
(B) $1\frac{1}{2}$
(C) 2
(D) $2\frac{1}{2}$
$\boxed{\text{(E)}}$ 3

$\boxed{\text{SOLN}}$ $$\frac{V_{\text{cup}}}{V_{\text{cone}}} = \frac{2\pi R^3}{\frac{2}{3}\pi R^3} = 3$$

□

27. When the sphere is submerged in the filled cup, water overflows. How many cones-full of water remain in the cup?

$\boxed{\text{(A)}}$ 1
(B) $1\frac{1}{2}$
(C) 2
(D) $2\frac{1}{2}$
(E) 3

$\boxed{\text{SOLN}}$

$$V_{\text{cup}} - V_{\text{sphere}} = 2\pi R^3 - \frac{4}{3}\pi R^3 = \frac{2}{3}\pi R^3$$

□

28. Miss Brown said, "I have many brothers and sisters. I am the sixth child and the number of my brothers is at least as large as the number of my sisters." Her younger brother added, "And I have at least twice as many sisters as brothers." How many siblings are there in the Brown family?

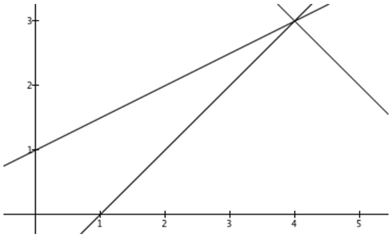
(A) 4

(B) 6

$\boxed{\text{(C)}}$ 7

(D) 8

(E) 9



$\boxed{\textit{SOLN}}$ Let g represent the girls and b the boys. Then the statements give us the following system of inequalities: $g+b \geq 7$, $b \geq g-1$, and $g \geq 2(b-1)$. This system of inequalities has an intersection of one point: (4, 3). Therefore, there are 4 girls and 3 boys. □

29. Which is equivalent to $\tan^2 x \cdot \sin^2 x$?

$\boxed{\text{(A)}}$ $\tan^2 x - \sin^2 x$

(B) $\cot^2 x \cdot \cos^2 x$

(C) $\cos^2 x$

(D) $\sec^2 x$

(E) $\sec^2 x - 1$

$\boxed{\textit{SOLN}}$ $\frac{\sin^2 x}{\cos^2 x} \cdot \sin^2 x = \frac{1-\cos^2 x}{\cos^2 x} \cdot \sin^2 x = \frac{\sin^2 x}{\cos^2 x} - \sin^2 x$ □

30. Manu arrives at the bus stop each day randomly between 7:42 AM and 7:51 AM. The bus arrives at the stop randomly between 7:48 AM and 7:52 AM and immediately departs. How likely is Manu to miss the bus?

$\boxed{\text{(A)}}$ Manu misses the bus $\frac{1}{8}$ of the time.

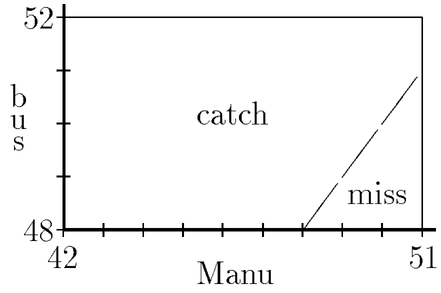
(B) Manu misses the bus $\frac{3}{16}$ of the time.

(C) Manu misses the bus $\frac{1}{3}$ of the time.

(D) Manu misses the bus $\frac{7}{50}$ of the time.

(E) Manu misses the bus $\frac{1}{2}$ of the time.

$\boxed{\textit{SOLN}}$ The rectangle represents the universe of possible arrival times. Use ratios of areas to get the probability. $P(\text{miss bus}) = \frac{\frac{1}{2}\cdot 3\cdot 3}{4\cdot 9} = \frac{\frac{1}{2}}{4} = \frac{1}{8}$.



□

31. In a group of 22 students, 12 like to play basketball, 15 like to play soccer, but two don't like to play either sport. How many like to play both basketball and soccer?

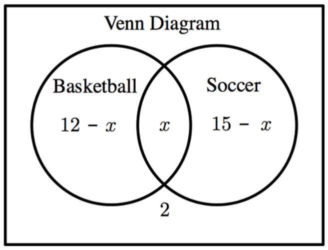
$\boxed{\text{(A)}}$ 7

(B) 27

(C) 3

(D) 5

(E) 9



$\boxed{\textit{SOLN}}$ $(12-x)+x+(15-x)+2 = 22 \implies 29 - x = 22 \implies x = 7$ □

32. Sum all prime numbers between 1 and 100 that are both 1 greater than a multiple of 4 and 1 less than a multiple of 5.

$\boxed{\text{(A)}}$ 118

(B) 137

(C) 158

(D) 187

(E) 245

$\boxed{\mathcal{SOLN}}$ For a prime to be 1 less than a multiple of 5 it must end in 9. There are only two primes in the given ranges that end in 9 and are 1 greater than a multiple of 4: 29, 89. □

33. If $a^x = c^q$ and $c^y = a^z$ (with $a, c > 0$ and $a, c \neq 1$), then which one of the following is true for all q, x, y, z?

$\boxed{\text{(A)}}$ $xy = qz$

(B) $\frac{x}{y} = \frac{q}{z}$

(C) $x + y = q + z$

(D) $x - y = q - z$

(E) $x + q = y + z$

$\boxed{\mathcal{SOLN}}$ Take $\log_a$ of both equations.

$$x = q \cdot \log_a c \Rightarrow \frac{x}{q} = \log_a c$$

$$z = y \cdot \log_a c \Rightarrow \frac{z}{y} = \log_a c$$

$$\frac{x}{q} = \frac{z}{y} \Rightarrow xy = qz$$

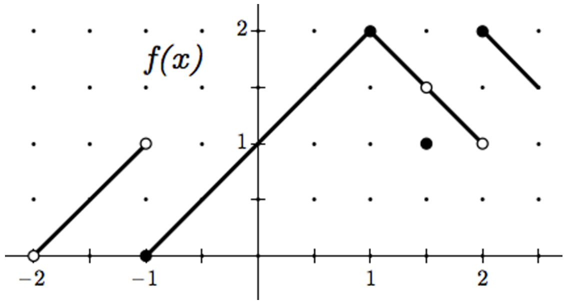
Or

$$(a^x)^z = a^{xz} = (a^z)^x = (c^y)^x = c^{xy}$$

$$\text{and} \quad (a^x)^z = (c^q)^z = c^{qz}$$

□

34. Select the **one false** statement about $f(x)$.



(A) $f(-1) = 0$

(B) $\lim_{x\to 1} f(x) = 2$

(C) $\lim_{x\to 2} f(x)$ does not exist.

(D) $\lim_{x\to -1^-} f(x) = 1$

$\boxed{\text{(E)}}$ $\lim_{x\to 3/2} f(x) = 1$

$\boxed{\mathcal{SOLN}}$ A–D true, but $\lim_{x\to 3/2} f(x) = \frac{3}{2}$ □

35. Students in a class are selected at random, one after the other, from a class consisting of 3 boys and 4 girls. What is the probability that girls and boys in the class alternate starting with a girl first?

$\boxed{\text{(A)}}$ $\frac{1}{35}$

(B) $\frac{34}{35}$

(C) $\frac{5}{7}$

(D) $\frac{6}{7}$

(E) $\frac{32}{35}$

$\boxed{\mathcal{SOLN}}$ The total number of ways to choose 7 students is $7! = 5040$. The number of ways to choose the alternating sequence is $4\cdot 3\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1 = 144$. The probability is $\frac{144}{5040} = \frac{1}{35}$.
Or cancel 4s, 3s, and 2s in

$$\begin{array}{ccccccccc} \text{G} & \text{B} & \text{G} & \text{B} & \text{G} & \text{B} & \text{G} & & \\ \hline \frac{4}{7} & \frac{3}{6} & \frac{3}{5} & \frac{2}{4} & \frac{2}{3} & \frac{1}{2} & \frac{1}{1} & = & \frac{1}{35} \end{array}$$

□

36. Two ladders that are resting on the floor can be made to reach the same vertical height on the wall. One of the ladders is twice as long as the other. If the shorter ladder makes an angle of 60° with the floor, what angle does the longer ladder make with the floor?

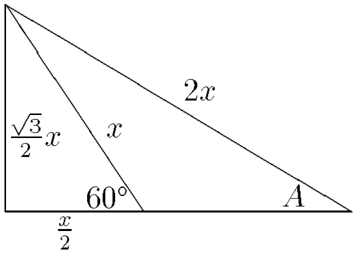
(A) $\sin^{-1} \frac{\sqrt{2}}{2}$

(B) $\sin^{-1} 2$

(C) $\sin^{-1} \sqrt{3}$

$\boxed{\text{(D)}}$ $\sin^{-1} \frac{\sqrt{3}}{4}$

(E) $\sin^{-1} \frac{1}{4}$



$\boxed{\text{SOLN}}$ $\sin A = \frac{\frac{\sqrt{3}}{2}x}{2x} = \frac{\sqrt{3}}{4}$ □

37. Cy's company was losing money. As a result Cy received a 25% pay cut. By what percentage must his new salary be raised to bring it back to the original level?

(A) 25%

$\boxed{\text{(B)}}$ $33\frac{1}{3}\%$

(C) 40%

(D) 50%

(E) 100%

$\boxed{\text{SOLN}}$ Let x be Cy's old pay rate and y his new rate. $y = \frac{3}{4}x$ so $x = \frac{4}{3}y$ □

38. What are the last two digits of 11^{22}?
Hint: $11^{22} = (10+1)^{22}$.

(A) 01

$\boxed{\text{(B)}}$ 21

(C) 33

(D) 51

(E) 81

$\boxed{\text{SOLN}}$ By the binomial theorem the first 21 terms of $11^{22} = (10+1)^{22}$ are divisible by 100, and so don't affect the last two digits. The last two terms are $\binom{22}{21} \times 10$ and 1. Since $\binom{22}{21} \times 10 = 220$ the last two digits are 21.
11^n always ends in 1. The penultimate digit is the last digit of n. □

39. The number represented as 256 in base 10 has what base 5 representation?

(A) 128

(B) 211

(C) 310

(D) 512

$\boxed{\text{(E)}}$ 2011

$\boxed{\text{SOLN}}$ $256_{10} = 2\cdot 5^3 + 0\cdot 5^2 + 1\cdot 5^1 + 1\cdot 5^0 = 2011_5$ □

40. If $z = 2\cos\frac{\pi}{12} + 2\,\mathrm{i}\,\sin\frac{\pi}{12}$, what is z^4?

(A) 16

$\boxed{\text{(B)}}$ $8 + 8\sqrt{3}\mathrm{i}$

(C) $8\sqrt{3} + 8\mathrm{i}$

(D) $\frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2}\mathrm{i}$

(E) $\left(\frac{13}{4} + \sqrt{3}\right) + \left(\frac{13}{4} - \sqrt{3}\right)\mathrm{i}$

$\boxed{\text{SOLN}}$ $z^4 = (2e^{\mathrm{i}\pi/12})^4 = 16e^{\mathrm{i}\pi/3} = 16\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) = 8 + 8\sqrt{3}\mathrm{i}.$ □