Snow College Mathematics Contest

April 5, 2011 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. What is the minimum number of times the graph of a fifth degree polynomial must cross the x-axis?

   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) 5

2. How many numbers from 1 to 2011 are divisible by either 20 or 11?

   (A) 282
   (B) 273
   (C) 291
   (D) 275
   (E) 279

3. What is the sum of the first seven cubes, $1^3 + 2^3 + 3^3 + \ldots + 7^3$?

   (A) 15^2
   (B) 21^2
   (C) 28^2
   (D) 36^2
   (E) 45^2

4. The continued fraction expression for δ (the "silver ratio") is

$$\delta = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

   Find a closed form expression for δ.

   (A) $1 + \sqrt{2}$
   (B) $\frac{1-\sqrt{2}}{2}$
   (C) $\frac{1+\sqrt{2}}{2}$
   (D) $\frac{\sqrt{2}}{2}$
   (E) $\frac{2}{\sqrt{2}}$

5. The expression

$$\frac{x}{1 - x - x^2}$$

   is a *generating function* for a famous sequence. Find the sequence by looking at the coefficients of the long division.

   (A) Harmonic sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
   (B) Primes: $2, 3, 5, 7, 11, 13, \ldots$
   (C) Squares: $1, 2, 4, 9, 16, 25, \ldots$
   (D) Fibonacci sequence: $1, 1, 2, 3, 5, 8, \ldots$
   (E) Triangular numbers: $1, 3, 6, 10, \ldots$

6. The *cardinality* (measure of the number of elements) of a set A is denoted $|A|$. Two sets have the same cardinality if there is a one-to-one correspondence between them. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ be the integers, $\mathbb{Q}$ be the rational numbers, $\mathbb{R}$ be the reals, and $\mathbb{C}$ be the complex numbers. Which statement is **not** true?

(A) $|\mathbb{R}| > |\mathbb{Z}|$

(B) $|\mathbb{Q}| = |\mathbb{N}|$

(C) $|\mathbb{C}| > |\mathbb{R}|$

(D) $|\mathbb{N}| = |\mathbb{Z}|$

(E) $|\mathbb{C}| > |\mathbb{Q}|$

7. What is the multiplicative inverse (reciprocal) of the complex number $a + b\,\mathrm{i}$?

(A) $a - b\,\mathrm{i}$

(B) $\frac{a-b\,\mathrm{i}}{a^2+b^2}$

(C) $-a - b\,\mathrm{i}$

(D) $\frac{1}{a} + \frac{1}{b}\,\mathrm{i}$

(E) $a^2 - b^2\,\mathrm{i}$

8. A popular dice game is called craps. In it you roll two standard six-sided dice and add the numbers showing on the top faces. What is the probability of rolling a sum of either 7 or 11 (called "throwing craps")?

(A) $\frac{2}{11}$

(B) $\frac{2}{9}$

(C) $\frac{1}{6}$

(D) $\frac{7}{36}$

(E) $\frac{6}{7}$

9. What is the value of $\mathrm{e}^{\mathrm{i}\pi} + 1$?

(A) -1

(B) 0

(C) 1

(D) π

(E) $\sqrt{2}$

10. Which of the following sets of data does **not** determine the relative shape of a triangle?

(A) the ratio of two sides and the included angle

(B) the ratios of the three altitudes

(C) the ratios of the three medians

(D) the ratio of the altitude to the corresponding base

(E) two angles

11. Goldbach's conjecture (still an open question) says that every even integer greater than 2 is the sum of two primes. How many different ways can this be done for 24?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

12. Simplify the expression for $\theta \neq n\pi$, $n \in \mathbb{Z}$.

$$\frac{\tan\theta - \sin\theta\cos\theta}{\tan\theta}$$

(A) $\sin\theta$

(B) $\cos\theta$

(C) $\sin^2\theta$

(D) $\cos^2\theta$

(E) 1

13. What is the least number of prime factors (not necessarily different) that 350 must be multiplied by so that the product is a perfect cube?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

14. What is the equation of the line perpendicular to $y = -\frac{1}{2}x + 4$ and passes through through the point $(2, 6)$?

(A) $y = 2x + 10$

(B) $y = \frac{1}{2}x + 5$

(C) $y = x + 4$

(D) $y = 2x + 2$

(E) $y = \frac{1}{2}x + \frac{1}{4}$

15. How many equilateral triangles (of all sizes) are there in the figure?

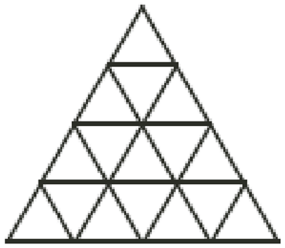
(A) 16

(B) 20

(C) 26

(D) 27

(E) 32



16. What is the perimeter of triangle ABC?

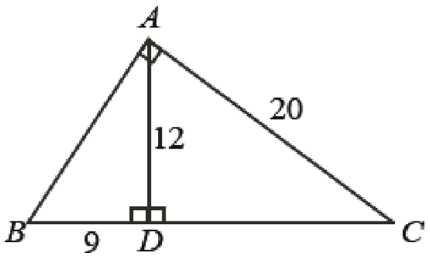
(A) 60

(B) 56

(C) 44

(D) 54

(E) 69



17. How many of the integers from 1 to 100 inclusive do **NOT** contain the digit 7?

(A) 19

(B) 20

(C) 80

(D) 81

(E) 90

18. What is the sum of the following?

$$2 - 1 + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \dots$$

(A) $\frac{4}{3}$

(B) $\frac{3}{2}$

(C) 2

(D) ∞

(E) $\frac{7}{4}$

19. A rope hangs from the top of a pole with 3 ft of it lying on the ground. When it is tightly stretched so that its end just touches the ground the end is 8 ft from the base of the pole. How long is the rope?

(A) 11 ft

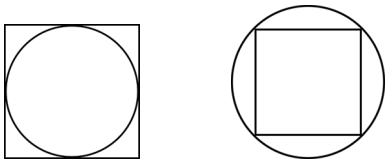
(B) 12 ft

(C) 13 ft

(D) $\frac{55}{6}$ ft

(E) $\frac{73}{6}$ ft

20. Which is a better (tighter) fit: a round peg in a square hole, or a square peg in a round hole? (A tighter fit fills up more of the hole.)



(A) round peg in a square hole

(B) square peg in a round hole

(C) both are equally tight

(D) need to know the length of the side of the square

(E) need to know the radius of the circle

21. Ancient Greeks classified the brightest stars as first magnitude and so on until the dimmest they could see with the naked eye were sixth magnitude. Consider two stars, labeled 1 and 2, with apparent magnitudes m_1 and m_2 and brightnesses b_1 and b_2, respectively. The *ratio* of the apparent brightnesses b_1/b_2 corresponds to a *difference* in the apparent magnitudes $(m_2 - m_1)$.

$$m_2 - m_1 = 2.5 \log_{10}\left(\frac{b_1}{b_2}\right)$$

If $m_2 = 22$, $m_1 = 2$ how much brighter does star 1 appear than star 2; what is b_1/b_2?

(A) 10^8

(B) 2.5

(C) 400

(D) 4

(E) $\frac{1}{4}$

22. Evaluate: $(-125)^{-2/3}$

(A) -25

(B) $-\frac{1}{25}$

(C) 25

(D) $\frac{1}{25}$

(E) $-\frac{1}{5}$

23. Say you buy 100 pounds of watermelon for a picnic. The melons are 99% water. By the date of the picnic, they dry out to 98% water. How much do they weigh now?

(A) 98 pounds

(B) 96 pounds

(C) 90 pounds

(D) 80 pounds

(E) 50 pounds

24. Flip a fair coin. Go 2 for heads, 1 for tails.

Start		Go back 2 spaces	End

If the probability of reaching End in exactly 4 turns is $\frac{2}{16}$, in exactly 5 turns is $\frac{3}{32}$, and in exactly 6 turns is $\frac{5}{64}$, what is the probability of reaching End in exactly 7 turns?

(A) $\frac{8}{128}$

(B) $\frac{5}{32}$

(C) $\frac{13}{256}$

(D) $\frac{1}{8}$

(E) $\frac{1}{4}$

25. The sum of the interior angles of a triangle on a sphere add up to more than π rad by an amount e called the *spherical excess*. The area of a spherical triangle is given by

$$A_\triangle = \frac{e}{4\pi} A_{\text{sphere}}$$

How much of a sphere does a spherical triangle with three right angles cover?

(A) $\frac{1}{8}$

(B) $\frac{1}{4\pi}$

(C) $\frac{1}{4}$

(D) $\frac{3}{8}$

(E) $\frac{\pi}{4}$

26. A circle of radius 2 and center E is inscribed inside square $\square$ABCD. Find the area that is inside $\triangle$AEB but outside the circle.

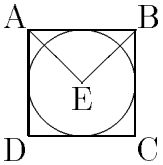
(A) $\pi - 3$

(B) $\frac{\pi}{2} - 1$

(C) $4 - \pi$

(D) $\pi - 2$

(E) $3 - \frac{\pi}{2}$



27. Of three boxes, one contains only apples, one contains only oranges, and one contains both apples and oranges. The boxes have been incorrectly labeled such that no label identifies the actual contents of the box it labels. Opening just one box, and without looking in the box, you take out one piece of fruit. By looking at the fruit, you can immediately label all of the boxes correctly. Which box do you open?

(A) the one labeled "apples"

(B) the one labeled "oranges"

(C) the one labeled "apples and oranges"

(D) either "apples" or "oranges" will work

(E) any of the boxes will work

28. Let $P(x) = x^3 - 2x^2 + 3x - 4$. Find the largest prime factor of $P(4) - P(2)$.

(A) 17

(B) 19

(C) 23

(D) 29

(E) 31

29. The unique solution to $ax + b = 10$ is $x = 2$; the unique solution to $bx + a = 8$ is $x = 3$. Find $a + b$.

(A) $\frac{26}{5}$

(B) $\frac{28}{5}$

(C) 6

(D) $\frac{32}{5}$

(E) $\frac{34}{5}$

30. Which polar equation describes the graph?

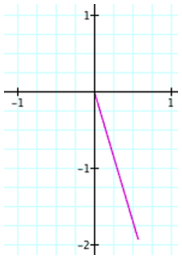
(A) $r = \theta$

(B) $r = 5\pi$

(C) $r = \tan^{-1}\frac{y}{x}$

(D) $r = \sin 5\theta$

(E) $\theta = 5$



31. The Pauli spin matrices σ_1, σ_2, and σ_3 appear in quantum mechanics. They are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

What is $-\mathrm{i}\sigma_1\sigma_2\sigma_3$?

(A) 0

(B) -1

(C) i

(D) $-\mathrm{i}$

(E) I

32. For your vacation you will travel first to New York City, then to London. You may travel to NYC by car, train, bus, or plane, and from NYC to London by ship or plane. How many different routes are possible?

(A) 6

(B) 8

(C) 10

(D) 12

(E) 14

33. In Mrs. Austen's 3rd grade class there are 25 students total. Of those 25, 15 like Oreos, 9 like Fudge Stripes cookies, and 4 students don't like either. Determine the probability of choosing a student who likes both Oreos and Fudge Stripes cookies.

(A) $\frac{3}{25}$

(B) $\frac{6}{25}$

(C) $\frac{19}{25}$

(D) $\frac{21}{25}$

(E) $\frac{24}{25}$

34. What is the value of the following product?

$$\tan 5° \times \tan 15° \times \tan 25° \times \tan 35° \times \tan 45° \times \tan 55° \times \tan 65° \times \tan 75° \times \tan 85°$$

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{3}}{3}$

(C) $\frac{\sqrt{3}}{2}$

(D) 1

(E) $\frac{\sqrt{2}}{2}$

35. Let R be the region below the curve $y = \sqrt{\sin(2x)}$ and above the x-axis with $0 < x < \pi/2$. Find the volume of the shape generated by revolving R around the x-axis.

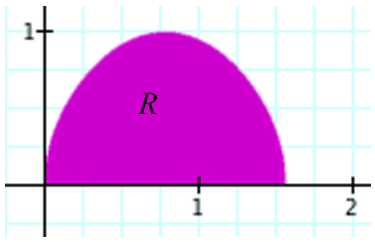
(A) 1

(B) π

(C) $\frac{2\pi}{3}$

(D) 1

(E) $\frac{\sqrt{2}}{2}$



36. All numbers in this question are in base four. What is 23^2?

(A) 1121

(B) 1033

(C) 1031

(D) 2311

(E) 1321

37. At Chicken Littles you can order boxes of 6, 9, or 20 chicken fingers. What is the sum of the digits in the largest number of fingers you *cannot* order? (E.g., if 21 then $2 + 1 = 3$.)

(A) 5

(B) 7

(C) 8

(D) 10

(E) 13

38. A large billiard table is 6 feet by 10 feet, and the cue ball moves as indicated. Into which hole will the cue ball fall?

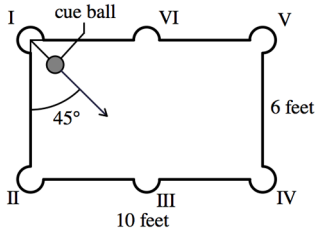
(A) II

(B) III

(C) IV

(D) I or V

(E) VI



39. Jack can mow the lawn in 2 hours. His sister Jill can mow the same lawn in 3 hours. If they use two mowers and work together how fast can they mow the lawn?

(A) 1 hour and 10 minutes

(B) 1 hour and 30 minutes

(C) 1 hour and 15 minutes

(D) 1 hour and 12 minutes

(E) 1 hour and 20 minutes

40. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. If we assume that the rate of the reaction $v = \frac{dx}{dt}$ is proportional to both the amount of the product and to the amount of the original substance present then we write

$$v = kx(a - x)$$

where x is the amount of product, a is the amount of substance at the beginning, and k is a positive constant. What is $v_{\max}$?

(A) $\frac{a}{2}$

(B) $kax - kx^2$

(C) ka^2

(D) $\frac{ka^2}{4}$

(E) $\frac{k^2a}{2}$