Snow College Mathematics Contest

key

April 6, 2010 **Senior division: grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. If f is a function such that $f(3) = 2$, $f(4) = 2$, and $f(n+4) = f(n+3) \cdot f(n+2)$ for all integers $n \geq 0$, what is the value of $f(6)$?

   (A) 4

   (B) 5

   (C) 6

   (D) 8

   (E) Not enough information

   SOLN $f(6) = f(2+4) = f(5) \cdot f(4) = (f(4) \cdot f(3)) \cdot f(4) = 2^2 \cdot 2 = 8$ □

2. What is the sum of the exponents in the prime factorization of 2010?

   (A) 2

   (B) 3

   (C) 4

   (D) 7

   (E) 2010

   SOLN $2010 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 67^1$. □

3. The integers 1 through 18 are paired up (each number used once) in such a way that the sum of each pair is a perfect square. What is 1's partner?

   (A) 1

   (B) 3

   (C) 8

   (D) 15

   (E) 16

   SOLN The perfect squares available are 4, 9, 16, and 25, so 1 could be paired with 3, 8, or 15. But 8 must pair with 17 (because 17 can't pair with anything else), and 3 must pair with 13 (if 3 pairs with 6, then 10 must pair with 15 and that doesn't leave anyone for 1). So 1 must pair with 15. □

4. Three boys were playing online. Al won twice as many games as Bill and Cy combined. Cy won twice as many games as Bill and eight fewer than Al. What is the combined total wins of the three boys?

   (A) 4

   (B) 12

   (C) 18

   (D) 24

   (E) 36

   SOLN Translate from English to a system of three linear equations.

$$\begin{aligned} A &= 2(B+C) \\ C &= 2B \\ C &= A-8 \end{aligned}$$

   The solution of the system is $A = 12, B = 4, C = 2$ and the sum is 18. □

5. What is the radius of the circle inscribed in a 3-4-5 right triangle?

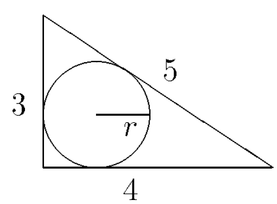
   (A) 1

   (B) 2

   (C) 2.5

   (D) 3

   (E) 3.5



   SOLN In general, the radius of a circle inscribed in a triangle is the area of the triangle divided by the semiperimeter $s = \frac{1}{2}(a+b+c)$.

$$r = \frac{A}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

   In the case of our right triangle

$$r = \frac{\frac{1}{2}ab}{\frac{1}{2}(a+b+c)} = \frac{3 \cdot 4}{3+4+5}$$

   However, it is intended that mere inspection would suffice to eliminate all but (A). □

6. Simplify $\cot\theta(\tan\theta + \sin\theta)$ for $\theta \neq \frac{n\pi}{2}$.

   (A) $1 + \cos\theta$

   (B) $\sin\theta + \cos\theta$

   (C) $\cos\theta + \cot\theta$

   (D) $\cos\theta$

   (E) $\tan\theta + \cot\theta$

   SOLN $\cot\theta \cdot \tan\theta = 1$ for $\theta \neq \frac{n\pi}{2}$ □

7. Prisoners want to break out of jail while the guards are out for lunch for one hour and fifteen minutes. The door of the jail is controlled by six ON/OFF switches. The jailbreakers need to find the right combination of all six switches to open the door. Each try takes one minute. If they have to try all the combinations except the initial setting to find the correct one, how much time do they have to run after the door is open and before the guards sound the alarm?

   (A) 11 min

   (B) 12 min

   (C) 13 min

   (D) 14 min

   (E) 15 min

   SOLN There are $2^6 = 64$ combinations, but since they have an initial state the prisoners only need to try 63 combinations. They have $75-63 = 12$ minutes to run. □

8. Determine the remainder when $x^8 + x^4 + 1$ is divided by $x - 1$.

(A) -3

(B) -1

(C) 1

[(D)] 3

(E) None of these

[SOLN] Of course, one could perform long division to get the answer. Synthetic division is a quicker method. However, the remainder theorem says that the remainder upon division (by a linear factor) is equal to the functional value at that point. The linear factor $x - 1$ corresponds to the point $x = 1$ and $f(1) = 1 + 1 + 1 = 3$. $\therefore r = 3$. □

9. How many natural number factors does the number 3600 have?

[(A)] 45

(B) 36

(C) 47

(D) 40

(E) 43

[SOLN] Brute force is effective but very slow. The prime factorization is $3600 = 2^4 \cdot 3^2 \cdot 5^2$. The number of factors is the product of each exponent plus one, i.e., $(4+1)(2+1)(2+1) = 45$. Also note that 3600 is a perfect square so the number of factors must be odd, immediately eliminating responses 36 and 40. □

10. A soccer ball has 12 black regular pentagons and _____ white regular hexagons.

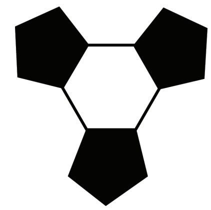
(A) 12

(B) 14

(C) 16

(D) 18

[(E)] 20



[SOLN] Every white hexagon abuts three black pentagons (as well as three other white hexagons). But each black pentagon abuts a white hexagon on each of its five sides. Therefore, there are $5 \times 12 = 60$ sides adjacent to white hexagons. $3x = 60 \quad \Rightarrow \quad x = 20$ □

11. What is the sum of the first twenty odd natural numbers, $1 + 3 + 5 + \ldots + 37 + 39$?

(A) 199

(B) 200

(C) 340

(D) 370

[(E)] 400

[SOLN] One method of solution is a Guass-type pairing:
$1 + 39 = 40$
$3 + 37 = 40$, etc.
There are 10 such pairs: $10 \cdot 40 = 400$. An even quicker way is so recall that the sum of the first n odd natural numbers is n^2 and $20^2 = 400$. □

12. Four black cows and three brown cows give as much milk in five days as three black cows and five brown cows give in four days. Which color cow is a better milker?

(A) brown

(B) black

(C) both are equal

(D) half and half

(E) Not enough information

SOLN Say a black cow gives x units of milk daily and a brown cow gives y units. Then $5(4x + 3y) = 4(3x + 5y)$. It seems that we don't have enough information to finish the problem (one equation and two unknowns), but we don't need to know x and y explicitly, we only need to know which is greater. Distribute and combine like terms to get $8x = 5y$, so brown cows give more milk. From Marilyn vos Savant. □

13. It can be argued that ϕ (the "golden ratio") is the "least irrational" number because its continued fraction expression has only ones.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$$

Find a closed form expression for ϕ.

(A) $\frac{1+\sqrt{5}}{2}$

(B) $\frac{1-\sqrt{2}}{5}$

(C) $\frac{1-\sqrt{3}}{2}$

(D) $\frac{\sqrt{5}}{2}$

(E) $\frac{\sqrt{5}}{3}$

SOLN See that the continued fraction expression is equivalent to $\phi = 1 + \frac{1}{\phi}$. Multiply by ϕ and rearrange to get $\phi^2 - \phi - 1 = 0$ and use the quadratic formula. The two solutions are $\frac{1\pm\sqrt{5}}{2}$, but only the positive one equals the original continued fraction. □

14. The three cube roots of 1 lie equally spaced around the unit circle in the complex plane. What are their values?

(A) $1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

(B) $-1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

(C) $-1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

(D) $1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

(E) $\mathrm{i},\ \frac{\sqrt{3}}{2}\mathrm{i} - \frac{1}{2},\ -\frac{\sqrt{3}}{2}\mathrm{i} - \frac{1}{2}$

SOLN Certainly one of the roots is 1; this eliminates choices B, C, E. In choice D they are all to the right of the y-axis, which wouldn't be equally spaced around the unit circle. □

15. A totally ordered set must obey the *trichotomy law*; that is, for any two elements a and b of the set, $a < b$, or $a > b$, or $a = b$. Which set is not totally ordered?

(A) $\mathbb{N}$, the natural numbers

(B) $\mathbb{Z}$, the integers

(C) $\mathbb{Q}$, the rational numbers

(D) $\mathbb{R}$, the real numbers

(E) $\mathbb{C}$, the complex numbers

SOLN For any two unequal complex numbers we cannot say one is greater than the other. (The modulus or norm or absolute value of a complex number, defined as $|z| = \sqrt{z\bar{z}}$, does allow comparison, but $|z| \in \mathbb{R}$.) □

16. Two objects are *homeomorphic* if one can be obtained from the other through a series of deformations (stretchings, twistings); however, tearing and gluing are not allowed. A circle is homeomorphic to an ellipse; but the letter "O" is not homeomorphic to the letter "Q" because of the tails. Which set of sans-serif letters is homeomorphic?

(A) {A, P, R}

(B) {E, F, Y}

(C) {E, H, K}

(D) {B, E, T}

(E) {C, G, X}

SOLN {E, F, Y} all have no holes and three tails. "T" is also homeomorphic to these three. {A, P, R} all have one hole, but "R" has two tails. "B" has two holes. "X" has four tails. □

17. If $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$, simplify the following.

$$\binom{n-1}{r-1} + \binom{n-1}{r}$$

(A) $\binom{n}{r}$

(B) $\binom{n-1}{2r-1}$

(C) $\binom{2n-2}{r-1}$

(D) $\binom{n-2}{r-1}$

(E) $\binom{n}{r} + \binom{n}{r}$

SOLN This can be proved algebraically from the definition given above, but a shorter way is to recognize that the rth entry in row n of Pascal's triangle is $\binom{n}{r}$. It is the sum of the two entries diagonally above it, which are $\binom{n-1}{r-1}$ and $\binom{n-1}{r}$. □

18. The kinetic energy of a moving object of mass m is $\frac{1}{2}mv^2$ where v is the speed of the object. Which of five identical cars gains the most kinetic energy?

(A) Car A goes from 0 mph to 25 mph.

(B) Car B goes from 10 mph to 30 mph.

(C) Car C goes from 20 mph to 35 mph.

(D) Car D goes from 40 mph to 50 mph.

(E) Car E goes from 60 mph to 65 mph.

SOLN Since m is the same for each car, we only need to compare $v_2^2 - v_1^2$.

$25^2 - 0^2 = 625$

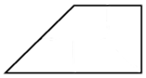
$30^2 - 10^2 = 800$

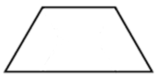
$35^2 - 20^2 = 825$

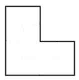
$50^2 - 40^2 = 900$

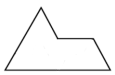
$65^2 - 60^2 = 625$ □

19. A *rep-tile* (replicating tile) of rep-n is a polygon that can be tiled with n smaller congruent copies of itself. Which of the following polygons is **not** a rep-4-tile?

(A) 

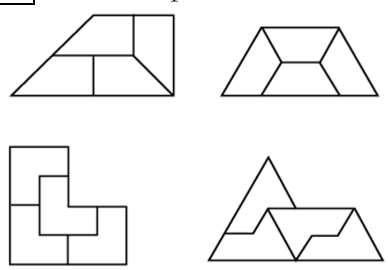
(B) 

(C) 

(D) 

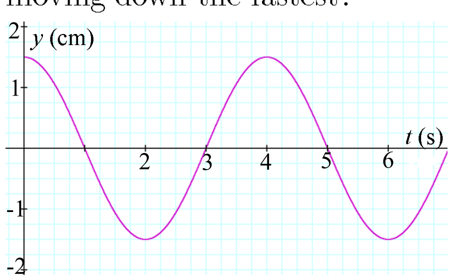
(E) 

SOLN Note: rep-tiles can tile the plane.



□

20. The position of an object oscillating vertically on a spring is shown. At what time is the object moving down the fastest?



(A) 2 s
(B) 3 s
(C) 4 s
(D) 5 s
(E) 5.5 s

SOLN The steepest negative slope is at $t = 5\,\text{s}$. □

21. Hyperbolic functions are so called because the point $(\cosh t, \sinh t)$ lies on the unit hyperbola—just as the point $(\cos t, \sin t)$ lies on the unit circle. Which of the following is a valid hyperbolic identity?

(A) $\cosh^2 t - \sinh^2 t = 1$
(B) $\cosh t + \sinh t = 1$
(C) $\cosh t - \sinh t = e^t$
(D) $\frac{\cosh t}{\sinh t} = \tanh t$
(E) $\cosh^2 t - \sinh^2 t = \cosh 2t$

SOLN Just as the parameterization of the unit circle $\cos t = x$ and $\sin t = y$ gives $x^2 + y^2 = 1$, the parameterization of the unit hyperbola $\cosh t = x$ and $\sinh t = y$ gives $x^2 - y^2 = 1$. (It is also true that $\cosh t + \sinh t = e^t$ and $\cosh^2 t + \sinh^2 t = \cosh 2t$ and $\frac{\sinh t}{\cosh t} = \tanh t$.) □

22. The number of odd numbers in row n of Pascal's triangle is 2^b where b is the number of ones in the binary (base two) expression of n. How many odd numbers are there in row 41 of Pascal's triangle?

(A) 2
(B) 3
(C) 4
(D) 8
(E) 16

1 — row 0
1 1 — row 1
1 2 1 — row 2
1 3 3 1 — row 3
1 4 6 4 1 — row 4
⋮

SOLN $41_{\text{ten}} = 101001_{\text{two}}$ which has 3 ones. So row 41 has $2^3 = 8$ odd numbers. □

23. What is the units digit of the integer 3^{2010}?

(A) 1

(B) 2

(C) 3

(D) 7

(E) 9

SOLN The sequence of units digits of 3^n is periodic with period four: $3, 9, 7, 1, \ldots$. Therefore the units digit of 3^{2010} is the digit in the 2010th term, which is 9. □

24. In the diagram, $ABCD$ is a square and P is a point on the circle with diameter CD, $CP = 7$, and $PD = 11$. What is the area of the square?

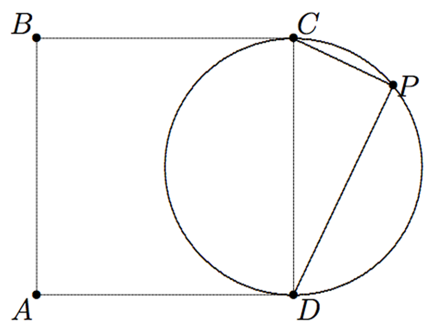
(A) 170

(B) 220

(C) 240

(D) 310

(E) 335



SOLN Angle CPD is a right angle. $CD = \sqrt{7^2 + 11^2} = \sqrt{49 + 121} = \sqrt{170}$, and $CD^2 = 170$. □

25. Given $\ln 1 = 0$, $\ln 5 = 1.6094$, and $\ln 2 = 0.6931$, what is the value of $\ln 0.2$?

(A) −0.6931

(B) 1.0644

(C) −1.6094

(D) 0.06931

(E) 0.9163

SOLN $\ln \frac{1}{x} = -\ln x$ □

26. What is the output of the following BASIC computer program?

```
10 dim F(7) :  rem dimension array
20 F(0) = 0
30 F(1) = 1 :  print F(1)
40 for i = 2 to 7
50 F(i) = F(i-2) + F(i-1)
60 print F(i)
70 next i
```

(A) 1 1 2 3 5 8 13

(B) 1 1 1 1 1 1

(C) 1 2 4 8 16 32

(D) 1 2 3 4 5 6 7

(E) 1 2 4 7 11 16 22

SOLN After the first two, each new number is the sum of the previous two. This is the Fibonacci sequence. □

27. A coin of diameter D is dropped randomly on a floor uniformly tiled with congruent squares of side L (with $L > D$). What is the probability that the coin will land entirely in one of the squares rather than on any tile boundaries?

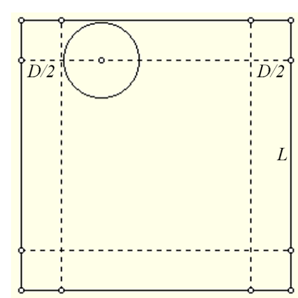
(A) $\pi D^2/L^2$

(B) πDL

(C) $\frac{3D}{(\pi L)^2}$

(D) $\frac{2L}{\pi D}$

(E) $\frac{(L-D)^2}{L^2}$



SOLN

The coin will land entirely in one square iff its center lands in a smaller square of side $L - D$ inside the tile. The probability is then just the ratio of the areas of the two squares. This is a precursor to Buffon's needle which can be used to calculate π. □

28. Consider the set of even integers, $2\mathbb{Z}$. If we define prime numbers as those positive numbers that can<u>not</u> be expressed as products of smaller positive elements of the set, what are the first four prime numbers in $2\mathbb{Z}$?

(A) 2, 3, 5, 7

(B) 2, 4, 6, 8

[(C)] 2, 6, 10, 14

(D) 0, 2, 4, 6

(E) 4, 6, 14, 18

[SOLN] 2 is prime. $4 = 2 \times 2$. 6 is prime. $8 = 2 \times 4$. 10 is prime. $12 = 2 \times 6$. The property of the uniqueness (up to ordering) of prime factorization does not hold in this ring; that is, the fundamental theorem of arithmetic does not apply to this system! For example, 2×18 and 6×6 are two distinct prime factorizations of 36 in $2\mathbb{Z}$. □

29. What is the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$?

(A) $\frac{4}{3}$

[(B)] $\frac{3}{2}$

(C) $\frac{5}{3}$

(D) 2

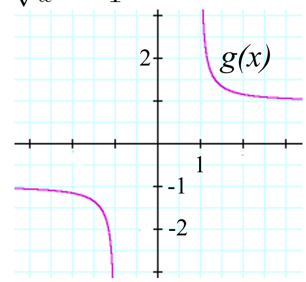
(E) ∞

[SOLN] The sum of $1+r+r^2+r^3+r^4+\ldots$ is $\frac{1}{1-r}$ for $|r| < 1$. In this case $r = \frac{1}{3}$ so the sum is $\frac{1}{1-(1/3)} = \frac{1}{2/3} = 3/2$.

Or, let $S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$. Then $3S = 3 + S \Rightarrow S = 3/2$. □

30. How many asymptotes does the function $g(x)$ have?

$$g(x) = \frac{x}{\sqrt{x^2 - 1}}$$



(A) 0

(B) 1

(C) 2

(D) 3

[(E)] 4

[SOLN] They are $x = \pm 1$ and $y = \pm 1$. □

31. If $f(x) = 3x - 2$, find $f(f(f(3)))$.

(A) 19

[(B)] 55

(C) 75

(D) 107

(E) 163

[SOLN] $f(3) = 7$; $f(7) = 19$; $f(19) = 55$. □

32. Exponentiation is not associative and is interpreted top down.

$$a^{b^c} = a^{(b^c)} \neq (a^b)^c = a^{b\cdot c}$$

An upper bound for Skewes' number, which G. H. Hardy said in the '30s was "the largest number which has ever served any definite purpose in mathematics" (though it has long since lost that distinction), is given as

$$10^{10^{10^{34}}}$$

What is ten times the number above?

(A) $10^{10^{10^{35}}}$

(B) $10^{10^{11^{34}}}$

(C) $10^{11^{10^{34}}}$

(D) $11^{10^{10^{34}}}$

$\boxed{\text{(E)}}$ $10^{10^{10^{34}}+1}$

$\boxed{\text{SOLN}}$ To multiply by ten a number which is an exponent of ten, just add one to the exponent. □

33. If the natural numbers are arranged in the following pattern what is the 7th number (from the left) in the 10th row?

$$\begin{array}{ccccccccc}
1 & & 2 & & 5 & & 10 & & 17 \quad \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
4 & \leftarrow & 3 & & 6 & & 11 & & 18 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
9 & \leftarrow & 8 & \leftarrow & 7 & & 12 & & 19 \\
 & & & & & & \downarrow & & \downarrow \\
16 & \leftarrow & 15 & \leftarrow & 14 & \leftarrow & 13 & & 20 \\
 & & & & & & & & \downarrow \\
25 & \leftarrow & 24 & \leftarrow & 23 & \leftarrow & 22 & \leftarrow & 21
\end{array}$$

(A) 88

$\boxed{\text{(B)}}$ 94

(C) 96

(D) 97

(E) 107

$\boxed{\text{SOLN}}$ Note patterns like the perfect squares down the left. The number in the kth position of the nth row is

$$\begin{array}{ll} n^2 - k + 1 & \text{if} \quad k \leq n \\ (k-1)^2 + n & \text{if} \quad k \geq n \end{array}$$

Extra: show that these two expressions are equal when $n = k$. □

34. In the table the sum of each row, column, and diagonal is the same.
What is the value of $A + B + C + D$?

(A) 80

$\boxed{\text{(B)}}$ 64

(C) 96

(D) 60

(E) 72

A	4	B
10	16	22
C	28	D

$\boxed{\text{SOLN}}$ The middle row sums to $10+16+22 = 48$. Therefore row 1 and row 3 have to each add to 48. This yields $(A+4+B)+(C+28+D) = 48+48$. Solving yields $A+B+C+D = 64$. □

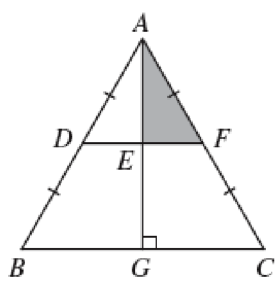
35. During the first three basketball games, Jordan scored an average of 18 points. After the fourth game, Jordan's scoring average dropped to 17 points. How many points did Jordan score in the fourth game?

(A) 14
(B) 15
(C) 16
(D) 17
(E) 18

SOLN With an average of 18 points per game for the first three games, Jordan scored a total of $3 \times 18 = 54$ points. Let x represent the number of points scored in game 4. $(54 + x)/4 = 17$. Solving for x yields $x = 14$. □

36. $\triangle ABC$ is isosceles with $AB = AC$, and $AG \perp BC$. Point D is the midpoint of AB, point F is the midpoint of AC, and E is the point of intersection of DF and AG. What fraction of the area of $\triangle ABC$ does the shaded area represent?

(A) 1/12
(B) 1/6
(C) 1/4
(D) 1/10
(E) 1/8



SOLN Let $x = EF$ and $y = AE$. Then the area of $\triangle AEF = \frac{1}{2}xy$.
By similar triangles, $GC = BG = 2x$, so $BC = 4x$. Similarly (good pun) $AG = 2y$. Therefore the area of $\triangle ABC$ is $\frac{1}{2}(4x)(2y) = 4xy$.

$$\frac{\frac{1}{2}xy}{4xy} = \frac{1}{8}$$ □

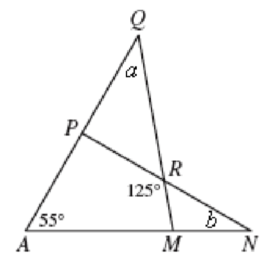
37. If $x + y = a$ and $x - y = b$, then what is the value of $2^{x^2-y^2}$?

(A) 2^{a+b}
(B) $2^{a^2-b^2}$
(C) 2^{a-b}
(D) $2^{a/b}$
(E) 2^{ab}

SOLN Factor $x^2 - y^2$ into $(x+y)(x-y)$; then $x^2 - y^2 = ab$. □

38. In the diagram, all lines that look straight are. What is the value of $a + b$?

(A) 55°
(B) 70°
(C) 75°
(D) 80°
(E) 90°



SOLN $m(\angle MRN) = 180° - 125° = 55°$ and $m(\angle PRQ) = 180° - 125° = 55°$. By the Exterior Angle Theorem, $m(\angle APR) = a + 55°$ and $m(\angle AMR) = b + 55°$. The sum of the interior angles of any quadrilateral $= 360°$, so $m(\angle PAM) + m(\angle APR) + m(\angle PRM) + m(\angle AMR) = 360°$. $\therefore$ $55° + (a + 55°) + 125° + (b + 55°) = 360°$. Solving for $a + b$ gives 70°.

Alternate solution: $m(\angle PRQ) = 55°$ Call $\angle APN = x$ and $\angle QPN = y$. Triangle Sum Theorem:

$$\begin{aligned} a + y + 55° &= 180° \\ b + x + 55° &= 180° \\ \hline a + b + x + y + 110° &= 360° \end{aligned}$$

$x + y = 180°$ so $a + b = 70°$ □

39. An 18-foot flagpole cracked in a violent storm and fell as if hinged. The tip of the pole hit the ground 12 feet from the base. How far up from the base was the crack?

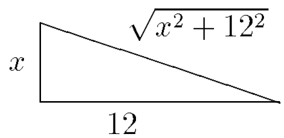
(A) 5 ft

(B) 6 ft

(C) 7 ft

(D) 8 ft

(E) 9 ft



SOLN The cracked pole forms the vertical side and hypotenuse of a right triangle.

$$x + \sqrt{x^2 + 12^2} = 18$$

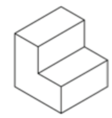
$$x^2 + 12^2 = (18 - x)^2$$

$$36x = 18^2 - 12^2$$

$$x = 3^2 - 2^2$$

Note the 5-12-13 right triangle. □

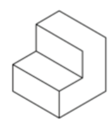
40.  is rotated to 

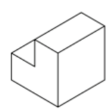
as  is rotated to

(A) 

(B) 

(C) 

(D) 

(E) 

SOLN The figure is rotated 90° cw looking down the positive z-axis. □