Snow College Mathematics Contest

April 6, 2010 **Senior division: grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. If f is a function such that $f(3) = 2$, $f(4) = 2$, and $f(n+4) = f(n+3) \cdot f(n+2)$ for all integers $n \geq 0$, what is the value of $f(6)$?

   (A) 4
   (B) 5
   (C) 6
   (D) 8
   (E) Not enough information

2. What is the sum of the exponents in the prime factorization of 2010?

   (A) 2
   (B) 3
   (C) 4
   (D) 7
   (E) 2010

3. The integers 1 through 18 are paired up (each number used once) in such a way that the sum of each pair is a perfect square. What is 1's partner?

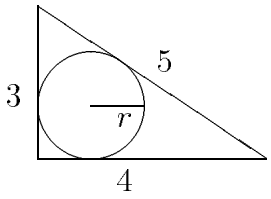
   (A) 1
   (B) 3
   (C) 8
   (D) 15
   (E) 16

4. Three boys were playing online. Al won twice as many games as Bill and Cy combined. Cy won twice as many games as Bill and eight fewer than Al. What is the combined total wins of the three boys?

   (A) 4
   (B) 12
   (C) 18
   (D) 24
   (E) 36

5. What is the radius of the circle inscribed in a 3-4-5 right triangle?

   (A) 1
   (B) 2
   (C) 2.5
   (D) 3
   (E) 3.5



6. Simplify $\cot\theta(\tan\theta + \sin\theta)$ for $\theta \neq \frac{n\pi}{2}$.

   (A) $1 + \cos\theta$
   (B) $\sin\theta + \cos\theta$
   (C) $\cos\theta + \cot\theta$
   (D) $\cos\theta$
   (E) $\tan\theta + \cot\theta$

7. Prisoners want to break out of jail while the guards are out for lunch for one hour and fifteen minutes. The door of the jail is controlled by six ON/OFF switches. The jailbreakers need to find the right combination of all six switches to open the door. Each try takes one minute. If they have to try all the combinations except the initial setting to find the correct one, how much time do they have to run after the door is open and before the guards sound the alarm?

   (A) 11 min
   (B) 12 min
   (C) 13 min
   (D) 14 min
   (E) 15 min

8. Determine the remainder when $x^8 + x^4 + 1$ is divided by $x - 1$.

   (A) -3
   (B) -1
   (C) 1
   (D) 3
   (E) None of these

9. How many natural number factors does the number 3600 have?

   (A) 45
   (B) 36
   (C) 47
   (D) 40
   (E) 43

10. A soccer ball has 12 black regular pentagons and _____ white regular hexagons.

    (A) 12
    (B) 14
    (C) 16
    (D) 18
    (E) 20

11. What is the sum of the first twenty odd natural numbers, $1 + 3 + 5 + \ldots + 37 + 39$?

    (A) 199
    (B) 200
    (C) 340
    (D) 370
    (E) 400

12. Four black cows and three brown cows give as much milk in five days as three black cows and five brown cows give in four days. Which color cow is a better milker?

    (A) brown
    (B) black
    (C) both are equal
    (D) half and half
    (E) Not enough information

13. It can be argued that ϕ (the "golden ratio") is the "least irrational" number because its continued fraction expression has only ones.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

    Find a closed form expression for ϕ.

    (A) $\frac{1+\sqrt{5}}{2}$
    (B) $\frac{1-\sqrt{2}}{5}$
    (C) $\frac{1-\sqrt{3}}{2}$
    (D) $\frac{\sqrt{5}}{2}$
    (E) $\frac{\sqrt{5}}{3}$

14. The three cube roots of 1 lie equally spaced around the unit circle in the complex plane. What are their values?

(A) $1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

(B) $-1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

(C) $-1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

(D) $1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

(E) $\mathrm{i},\ \frac{\sqrt{3}}{2}\mathrm{i}-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\mathrm{i}-\frac{1}{2}$

15. A totally ordered set must obey the *trichotomy law*; that is, for any two elements a and b of the set, $a < b$, or $a > b$, or $a = b$. Which set is not totally ordered?

(A) $\mathbb{N}$, the natural numbers

(B) $\mathbb{Z}$, the integers

(C) $\mathbb{Q}$, the rational numbers

(D) $\mathbb{R}$, the real numbers

(E) $\mathbb{C}$, the complex numbers

16. Two objects are *homeomorphic* if one can be obtained from the other through a series of deformations (stretchings, twistings); however, tearing and gluing are not allowed. A circle is homeomorphic to an ellipse; but the letter "O" is not homeomorphic to the letter "Q" because of the tails. Which set of sans-serif letters is homeomorphic?

(A) {A, P, R}

(B) {E, F, Y}

(C) {E, H, K}

(D) {B, E, T}

(E) {C, G, X}

17. If $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, simplify the following.

$$\binom{n-1}{r-1} + \binom{n-1}{r}$$

(A) $\binom{n}{r}$

(B) $\binom{n-1}{2r-1}$

(C) $\binom{2n-2}{r-1}$

(D) $\binom{n-2}{r-1}$

(E) $\binom{n}{r} + \binom{n}{r}$

18. The kinetic energy of a moving object of mass m is $\frac{1}{2}mv^2$ where v is the speed of the object. Which of five identical cars gains the most kinetic energy?

(A) Car A goes from 0 mph to 25 mph.

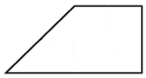
(B) Car B goes from 10 mph to 30 mph.

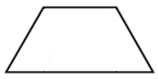
(C) Car C goes from 20 mph to 35 mph.

(D) Car D goes from 40 mph to 50 mph.

(E) Car E goes from 60 mph to 65 mph.

19. A *rep-tile* (replicating tile) of rep-n is a polygon that can be tiled with n smaller congruent copies of itself. Which of the following polygons is **not** a rep-4-tile?

(A) 

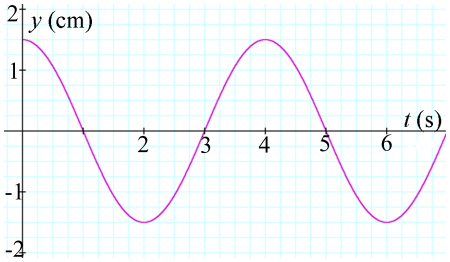
(B) 

(C) 

(D) 

(E) 

20. The position of an object oscillating vertically on a spring is shown. At what time is the object moving down the fastest?



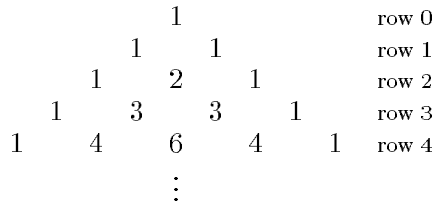
(A) 2 s
(B) 3 s
(C) 4 s
(D) 5 s
(E) 5.5 s

21. Hyperbolic functions are so called because the point $(\cosh t, \sinh t)$ lies on the unit hyperbola—just as the point $(\cos t, \sin t)$ lies on the unit circle. Which of the following is a valid hyperbolic identity?

(A) $\cosh^2 t - \sinh^2 t = 1$
(B) $\cosh t + \sinh t = 1$
(C) $\cosh t - \sinh t = \mathrm{e}^t$
(D) $\frac{\cosh t}{\sinh t} = \tanh t$
(E) $\cosh^2 t - \sinh^2 t = \cosh 2t$

22. The number of odd numbers in row n of Pascal's triangle is 2^b where b is the number of ones in the binary (base two) expression of n. How many odd numbers are there in row 41 of Pascal's triangle?

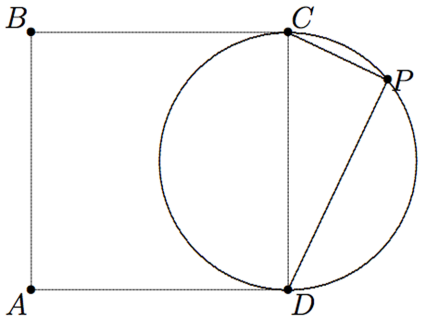
(A) 2
(B) 3
(C) 4
(D) 8
(E) 16



23. What is the units digit of the integer 3^{2010}?

(A) 1
(B) 2
(C) 3
(D) 7
(E) 9

24. In the diagram, $ABCD$ is a square and P is a point on the circle with diameter CD, $CP = 7$, and $PD = 11$. What is the area of the square?



(A) 170
(B) 220
(C) 240
(D) 310
(E) 335

25. Given $\ln 1 = 0$, $\ln 5 = 1.6094$, and $\ln 2 = 0.6931$, what is the value of $\ln 0.2$?

(A) -0.6931
(B) 1.0644
(C) -1.6094
(D) 0.06931
(E) 0.9163

26. What is the output of the following BASIC computer program?

```
10 dim F(7) :  rem dimension array
20 F(0) = 0
30 F(1) = 1 :  print F(1)
40 for i = 2 to 7
50 F(i) = F(i-2) + F(i-1)
60 print F(i)
70 next i
```

(A) 1 1 2 3 5 8 13

(B) 1 1 1 1 1 1

(C) 1 2 4 8 16 32

(D) 1 2 3 4 5 6 7

(E) 1 2 4 7 11 16 22

27. A coin of diameter D is dropped randomly on a floor uniformly tiled with congruent squares of side L (with $L > D$). What is the probability that the coin will land entirely in one of the squares rather than on any tile boundaries?

(A) $\pi D^2/L^2$

(B) πDL

(C) $\frac{3D}{(\pi L)^2}$

(D) $\frac{2L}{\pi D}$

(E) $\frac{(L-D)^2}{L^2}$

28. Consider the set of even integers, $2\mathbb{Z}$. If we define prime numbers as those positive numbers that cannot be expressed as products of smaller positive elements of the set, what are the first four prime numbers in $2\mathbb{Z}$?

(A) 2, 3, 5, 7

(B) 2, 4, 6, 8

(C) 2, 6, 10, 14

(D) 0, 2, 4, 6

(E) 4, 6, 14, 18

29. What is the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$?

(A) $\frac{4}{3}$

(B) $\frac{3}{2}$

(C) $\frac{5}{3}$

(D) 2

(E) ∞

30. How many asymptotes does the function $g(x)$ have?

$$g(x) = \frac{x}{\sqrt{x^2-1}}$$

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

31. If $f(x) = 3x - 2$, find $f(f(f(3)))$.

(A) 19

(B) 55

(C) 75

(D) 107

(E) 163

32. Exponentiation is not associative and is interpreted top down.

$$a^{b^c} = a^{(b^c)} \neq (a^b)^c = a^{b \cdot c}$$

An upper bound for Skewes' number, which G. H. Hardy said in the '30s was "the largest number which has ever served any definite purpose in mathematics" (though it has long since lost that distinction), is given as

$$10^{10^{10^{34}}}$$

What is ten times the number above?

(A) $10^{10^{10^{35}}}$

(B) $10^{10^{11^{34}}}$

(C) $10^{11^{10^{34}}}$

(D) $11^{10^{10^{34}}}$

(E) $10^{10^{10^{34}}+1}$

33. If the natural numbers are arranged in the following pattern what is the 7th number (from the left) in the 10th row?

1		2		5		10		17		⋮
		↓		↓		↓		↓		
4	←	3		6		11		18		
				↓		↓		↓		
9	←	8	←	7		12		19		
						↓		↓		
16	←	15	←	14	←	13		20		
								↓		
25	←	24	←	23	←	22	←	21		

(A) 88

(B) 94

(C) 96

(D) 97

(E) 107

34. In the table the sum of each row, column, and diagonal is the same.
What is the value of $A + B + C + D$?

A	4	B
10	16	22
C	28	D

(A) 80

(B) 64

(C) 96

(D) 60

(E) 72

35. During the first three basketball games, Jordan scored an average of 18 points. After the fourth game, Jordan's scoring average dropped to 17 points. How many points did Jordan score in the fourth game?

(A) 14

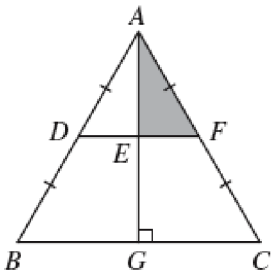
(B) 15

(C) 16

(D) 17

(E) 18

36. $\triangle ABC$ is isosceles with $AB = AC$, and $AG \perp BC$. Point D is the midpoint of AB, point F is the midpoint of AC, and E is the point of intersection of DF and AG. What fraction of the area of $\triangle ABC$ does the shaded area represent?



(A) 1/12

(B) 1/6

(C) 1/4

(D) 1/10

(E) 1/8

37. If $x + y = a$ and $x - y = b$, then what is the value of $2^{x^2 - y^2}$?

(A) 2^{a+b}

(B) $2^{a^2-b^2}$

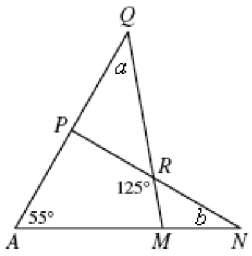
(C) 2^{a-b}

(D) $2^{a/b}$

(E) 2^{ab}

38. In the diagram, all lines that look straight are. What is the value of $a + b$?

(A) 55°
(B) 70°
(C) 75°
(D) 80°
(E) 90°



39. An 18-foot flagpole cracked in a violent storm and fell as if hinged. The tip of the pole hit the ground 12 feet from the base. How far up from the base was the crack?

(A) 5 ft
(B) 6 ft
(C) 7 ft
(D) 8 ft
(E) 9 ft

40.  is rotated to  as  is rotated to

(A) 

(B) 

(C) 

(D) 

(E)