Snow College Mathematics Contest

key

April 7, 2009 **Senior division: grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. Simplify $7^{2009} \pmod 6$.

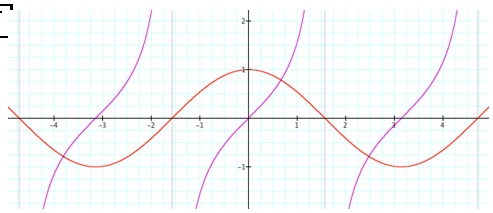
   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) 5

   SOLN $7 \pmod 6 = 1$ so $7^{2009} \pmod 6 = 1^{2009} \pmod 6 = 1$ □

2. What are the asymptotes of $y = \tan x$?

   (A) $y = n\pi$ (n an integer)
   (B) $x = n\pi$ (n an integer)
   (C) $x = n\frac{\pi}{2}$ (n an integer)
   (D) $y = (2n+1)\frac{\pi}{2}$ (n an integer)
   (E) $x = (2n+1)\frac{\pi}{2}$ (n an integer)

   SOLN

   $\tan x = \frac{\sin x}{\cos x}$ so $\tan x$ is undefined (and has vertical asymptotes) where $\cos x = 0$. □

3. If a steel ball one inch in diameter weighs one pound, how much will a steel ball two inches in diameter weigh?

   (A) 2 lbs
   (B) 3 lbs
   (C) 4 lbs
   (D) 5 lbs
   (E) 8 lbs

   SOLN The volume of a sphere scales as the radius cubed ($V = \frac{4}{3}\pi r^3$), so if the radius is doubled the volume is multiplied by $2^3 = 8$, and so is the weight if we assume constant density. □

4. In a bowl of red, green, blue, and yellow jelly beans, all but 16 are red, all but 16 are green, all but 16 are blue, and all but 18 are yellow. How many jelly beans are there?

   (A) 16
   (B) 18
   (C) 22
   (D) 48
   (E) 66

   SOLN Write a system of four equations in four unknowns; but rather than solving it, add up all the equations.

$$\begin{array}{rcrcrcrcr} & & G & + & B & + & Y & = & 16 \\ R & + & & & B & + & Y & = & 16 \\ R & + & G & + & & & Y & = & 16 \\ R & + & G & + & B & & & = & 18 \\ \hline 3R & + & 3G & + & 3B & + & 3Y & = & 66 \end{array}$$

   3 (all the beans) = 66 □

5. A stock loses 60% of its value. What must the percent *increase* be to recover all of its lost value?

(A) 60%

(B) 120%

[(C)] 150%

(D) 200%

(E) 250%

[SOLN] Losing 60% leaves 40% of the value. If x is the *increase* then we solve the equation $(0.40)(1 + x) = 1.00$. The solution is 1.5 or 150%. □

6. The sum $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 357° + \cos 358° + \cos 359°$ is equal to

(A) $\pi/2$

(B) π

(C) 0

(D) 1

[(E)] -1

[SOLN] Each cosine will pair up and cancel with another except $\cos 180°$.
$\cos 1° + \cos 179° = 0$,
$\cos 2° + \cos 178° = 0$, etc. □

7. Over the life of a car the odometer rolls from 000 000 to 999 999. How many times does the digit 1 appear in the odometer reading? (Example: the reading 131 518 has three occurrences.)

(A) 100 000

(B) 111 111

[(C)] 600 000

(D) 666 666

(E) 700 000

[SOLN] There are a million numbers between 000 000 and 999 999 inclusive, each with six digits, for a total of six million digits. Each of the ten digits appears as often as the others, so the digit 1 appears $6\,000\,000 \div 10 = 600\,000$ times, as does 2, 3, etc. □

8. When two congruent equilateral triangles share a common center, their union can be a star. If their intersection is a regular hexagon with an area of 60, what is the area of one of the original equilateral triangles?

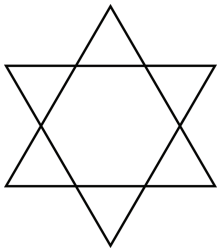
[(A)] 90

(B) 60

(C) 120

(D) 50

(E) 75



[SOLN] Fold the outer triangular flaps of one large triangle in so their vertices meet in the middle. The flaps add half again the area of the hexagon. □

9. What is the best parabolic approximation for $\cos x$ near $x = 0$?

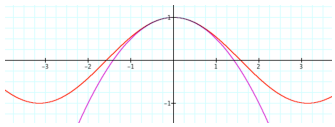
(A) x^2

(B) $-x^2 + 1$

(C) $x^2 + 1$

(D) $\frac{x^2}{2} + 1$

[(E)] $-\frac{x^2}{2} + 1$



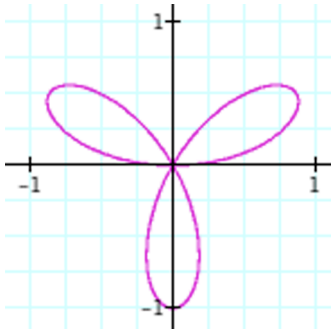
SOLN

Truncate the series expansion after the second term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

□

10. Which polar equation best represents the graph?



(A) $r = 3\theta$

[(B)] $r = \sin 3\theta$

(C) $r = \cos 3\theta$

(D) $r = 3\cos\theta$

(E) $r = \cos\theta$

SOLN The point $(r, \theta) = (0, 0)$ eliminates choices C, D, and E. $r = 3\theta$ is a spiral, so A is eliminated. Check other points to verify $r = \sin 3\theta$. □

11. Following the logic used in the first three squares, what is the missing number in the fourth square?

5	5
50	
2	

6	3
36	
2	

4	7
28	
1	

8	2
?	
3	

[(A)] 48

(B) 64

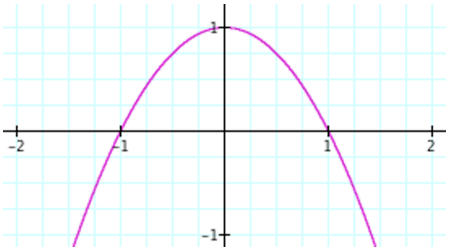
(C) 4

(D) 12

(E) 40

SOLN The central number is the product of the other three. □

12. What is the area bounded by the x-axis and the curve $y = -x^2 + 1$?



(A) 1

(B) $\frac{\pi}{3}$

[(C)] $\frac{4}{3}$

(D) $\frac{5}{3}$

(E) 2

SOLN Simple geometric estimation eliminates choices A and E.

$$\begin{aligned} \text{area} &= \int_{-1}^{1} (-x^2 + 1)\,dx \\ &= \left[x - \frac{1}{3}x^3\right]_{-1}^{1} \\ &= 1 - \frac{1}{3} + 1 - \frac{1}{3} \\ &= \frac{4}{3} \end{aligned}$$

(Graph made with Graphing Calculator.) □

13. An *equivalence relation* is a relation which is reflexive, symmetric, and transitive. Consider the relation $\mathcal{R}$:

$$x \mathcal{R} y \quad \text{if and only if} \quad x \geq y$$

Which statement is true?

(A) $\mathcal{R}$ is an equivalence relation.

(B) $\mathcal{R}$ is not an equivalence relation only because it is not reflexive.

[(C)] $\mathcal{R}$ is not an equivalence relation only because it is not symmetric.

(D) $\mathcal{R}$ is not an equivalence relation only because it is not transitive.

(E) $\mathcal{R}$ is not an equivalence relation because two criteria fail.

SOLN $\mathcal{R}$ is not symmetric; for example $(2, 1) \in \mathcal{R}$, but $(1, 2) \notin \mathcal{R}$. However, because it is reflexive, *anti*-symmetric, and transitive, it is a *partial order*. □

14. The *eigenvalues* of a matrix A are the values of λ which satisfy $\det(\lambda I - A) = 0$. What are the eigenvalues of the following matrix?

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

(A) $2, -3$

(B) $4, -2$

(C) $3, 10$

(D) $-2, 5$

(E) $-3, -10$

SOLN $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} =$
$\lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0$ □

15. The period T of a small-angle simple pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$ where g is the acceleration due to gravity and L is the length of the pendulum. By what factor does the period increase when the length is tripled?

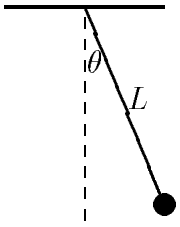
(A) 3

(B) $\sqrt{3}$

(C) $\sqrt{9.8}$

(D) 9

(E) $3\sqrt{2}$



SOLN $T' = 2\pi\sqrt{\frac{3L}{g}} = \sqrt{3}T$ □

16. A *fixed point* for a function $f(x)$ is a real number r such that $f(r) = r$. How many of the following classes of functions *must* have a fixed point?

- polynomial of odd degree > 1
- polynomial of even degree > 0
- trig function $y = A\sin Bx + C$
- rational function $y = (x - a)/(x - b)$

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

SOLN A fixed point must be where the function intersects the line $y = x$. All polynomials of odd degree > 1 and all functions $y = A\sin Bx + C$ must cross that line. However, $y = (x-2)/(x-1)$ and $y = x^2 + 1$ never do. □

17. A square is inscribed in a circle C_1. A second circle C_2 is inscribed in the square. What is the ratio of the areas of the circles, A_1/A_2?

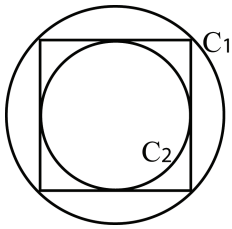
(A) $\sqrt{2}$

(B) π

(C) $\pi/2$

(D) $\pi\sqrt{2}$

(E) 2



SOLN Call the length of a side of the square s. The radius of C_2 is $s/2$, so $A_2 = \pi(s/2)^2 = \pi s^2/4$. The radius of C_1 is $\sqrt{2}\,s/2$, so $A_1 = \pi(\sqrt{2}\,s/2)^2 = \pi s^2/2$. Then $A_1/A_2 = 2$. □

18. Which of the following is the identity function $f(x) = x$ for *all* real numbers?

(A) $f(x) = \mathrm{e}^{\ln x}$

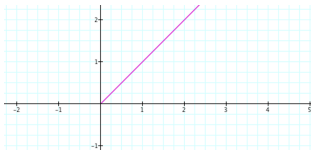
(B) $f(x) = \ln \mathrm{e}^{x}$

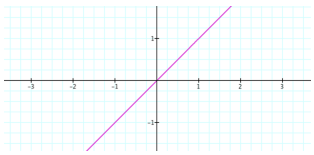
(C) $f(x) = \sin(\arcsin x)$

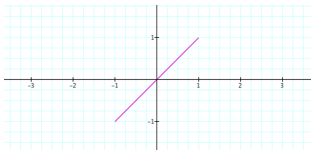
(D) $f(x) = \arctan(\tan x)$

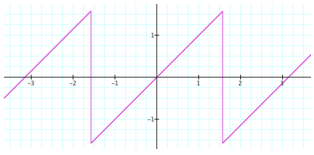
(E) $f(x) = \sqrt{x^2}$

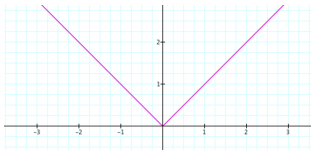
SOLN

(A) 

(B) 

(C) 

(D) 

(E) 

□

19. A *perfect number* is a positive integer which is the sum of its proper positive divisors (that is, excluding the number itself). For example, 28 is a perfect number because $1 + 2 + 4 + 7 + 14 = 28$.
What is the smallest perfect number?

(A) 2

(B) 4

(C) 6

(D) 12

(E) 14

SOLN $1 + 2 + 3 = 6$. We don't know of any odd perfect numbers. □

20. A general hypersphere in n dimensions is the locus of points equidistant r from a central point. In 2D space this is a circle whose circumference is $2\pi r$ and whose interior area is πr^2. In 3D space this is a sphere whose surface area is $4\pi r^2$ and whose interior volume is $\frac{4}{3}\pi r^3$. Find a pattern to deduce the hypersurface area of a 4D hypersphere given its interior hypervolume is $\frac{1}{2}\pi^2 r^4$.

(A) $2\pi^2 r^3$

(B) $8\pi^2 r^3$

(C) $8\pi r^3$

(D) $4\pi r^3$

(E) $2\pi r^3$

dim	surface	interior
1	2	$2r$
2	$2\pi r$	πr^2
3	$4\pi r^2$	$\frac{4}{3}\pi r^3$
4		$\frac{1}{2}\pi^2 r^4$
5	$\frac{8}{3}\pi^2 r^4$	$\frac{8}{15}\pi^2 r^5$

SOLN While it is true that

$$S_{n+2} = \frac{2\pi r^2}{n} S_n \quad \text{and} \quad V_{n+2} = \frac{2\pi r^2}{n+2} V_n$$

the easiest way is to see that

$$S_n = \frac{\mathrm{d}V_n}{\mathrm{d}r}$$

□

21. The Snow College Math Contest logo features the Gamma function which has the property $\Gamma(n+1) = n\Gamma(n)$ (for $n > 0$). Given $\Gamma(1) = 1$, what is $\Gamma(6)$?

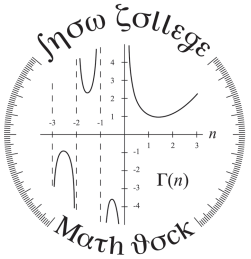
(A) 24

(B) 30

(C) 60

(D) 120

(E) 720



SOLN $\Gamma(n+1) = n!$ so $\Gamma(6) = 5!$ □

22. Which of the following is NOT a factor of $x^4 - 4x^3 - x^2 + 16x - 12$?

(A) $x - 2$

(B) $x + 2$

(C) $x - 1$

[(D)] $x + 1$

(E) $x - 3$

SOLN Long way: Pick any of the choices and divide it into the original polynomial. If it doesn't divide evenly you've found the answer; if it does then you've reduced the polynomial by one degree, then re-try with another of the choices.
$(x^4 - 4x^3 - x^2 + 16x - 12) \div (x - 2) = x^3 - 2x^2 - 5x + 6$
$(x^3 - 2x^2 - 5x + 6) \div (x + 2) = x^2 - 4x + 3$
$(x^2 - 4x + 3) \div (x - 1) = x - 3$
At this point we've found all the linear factors, so the remaining choice (D) must be the correct answer.
Short way: think of the polynomial as a function $f(x)$. If $(x - a)$ is a factor of the polynomial, then a is a zero of the function. Of the choices, it is easy to see that $f(-1) \neq 0$, so (D) is the correct choice. □

23. A distance runner can run 6 mph downhill, but his speed is cut to 2 mph when he runs uphill. If he runs 6 miles downhill and 6 miles uphill, what is his average speed?

(A) 2 mph

[(B)] 3 mph

(C) 4 mph

(D) 5 mph

(E) 6 mph

SOLN The average speed is the total distance divided by the total time. The time downhill is 1 h and the time uphill is 6 miles ÷ 2 mph = 3 h. So the total time is 4 h. 12 miles ÷ 4 h = 3 mph. □

24. The numbers 1, a, 9 form an arithmetic sequence. The numbers 1, b, 9 form a geometric sequence. What is $a + b$?

(A) 1

(B) 6

[(C)] 8

(D) 9

(E) 11

SOLN $a = 5$ and $b = 3$ so $a + b = 8$ □

25. What is the area of the largest triangle that can be inscribed in a semi-circle of radius 3?

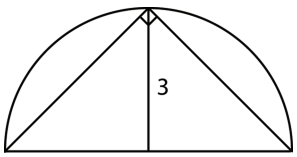
[(A)] 9

(B) 9.5

(C) 18

(D) 27

(E) 54



SOLN We want to maximize the base and the height. The largest base is when it is the diameter; the largest height is then with the third vertex at the top of the semi-circle. $A = \frac{1}{2}BH = \frac{1}{2}(6)(3) = 9$. By the way, any triangle inscribed in a semi-circle is guaranteed to be a right triangle. □

26. Simplify the inequality.

$$1 < \frac{1}{3-2x} < 3$$

(A) $-\frac{3}{4} < x < -\frac{3}{16}$
(B) $\frac{3}{4} < x < 3$
(C) $-\frac{16}{3} < x < -\frac{4}{3}$
(D) $1 < x < \frac{4}{3}$
(E) $-\frac{4}{3} < x < 1$

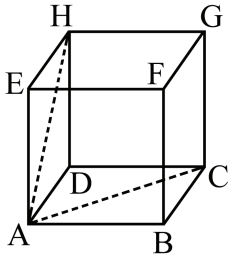
SOLN

$$1 < \frac{1}{3-2x} < 3$$
$$3-2x < 1 \quad \text{and} \quad 1 < 3(3-2x)$$
$$2 < 2x \quad \text{and} \quad -8 < -6x$$
$$1 < x \quad \text{and} \quad \frac{-8}{-6} > x$$
$$1 < x \quad \text{and} \quad x < \frac{4}{3}$$

□

27. Diagonals AC and AH are drawn in cube ABCDEFGH. What is the measure of $\angle$ CAH?

(A) 60°
(B) 72°
(C) 45°
(D) 90°
(E) 54°



SOLN Draw in a third diagonal HC and note from symmetry that AC, AH, and HC must be of equal length. These three diagonals form an equilateral triangle, so each angle measures 60°. □

28. Find the value of x in the continued fraction.

$$\frac{1}{x-\frac{1}{x-\frac{1}{x-\frac{1}{x-\cdots}}}} = 2$$

(A) 0.5
(B) 1
(C) 1.5
(D) 2
(E) 2.5

SOLN Since the fraction is continued, the equation is equal to

$$\frac{1}{x-2} = 2$$

Solving for x gives 2.5. □

29. Two standard dice are rolled twice. What is the probability of obtaining an even sum followed by another even sum?

(A) 0.25
(B) 0.45
(C) 0.5
(D) 0.75
(E) 0.866

SOLN Each sum must be even or odd, with equal probabilities (18/36 and 18/36), so there is a 0.5 probability of the first being even and the same 0.5 probability for the second being even. $0.5 \times 0.5 = 0.25$ □

30. What are the sum and product of the roots of $x^2 + 3x - 2 = 0$?

(A) 3 and 2

(B) 3 and -2

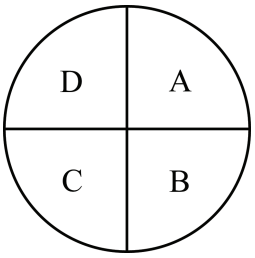
(C) -3 and 2

(D) -3 and -2

(E) None of these

SOLN Note that $(x-a)(x-b) = x^2 - (a+b)x + ab$. Comparing this with the given equation, we see that $-(a+b) = 3$ and $ab = -2$. (The roots are $\frac{-3 \pm \sqrt{17}}{2}$.) □

31. Gardens A and B each contain 50 plants. The average number of plants in Gardens B and C is 65. Which of the following is true if x represents the average number of plants in Gardens C and D?



(A) $x < 45$

(B) $x = 45$

(C) $45 < x < 46$

(D) $x \geq 46$

(E) Not enough information

SOLN $B = 50$

$$\frac{B+C}{2} = 65 \quad \Rightarrow \quad C = 80$$

$$x = \frac{C+D}{2} = \frac{80+D}{2}$$

Depending on D any of the choices (A), (B), (C), and (D) could be true, so (E) is correct. □

32. How long is an escalator if it makes a 30° angle with the floor and caries people a vertical distance of 20 ft between floors?

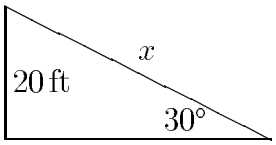
(A) 10 ft

(B) 20 ft

(C) 30 ft

(D) 40 ft

(E) 50 ft



SOLN From SOH CAH TOA

$$\sin 30° = \frac{20\,\text{ft}}{x} = \frac{1}{2}$$ □

33. Pascal's triangle has 1s at each end of each row and the other numbers being the sum of the two immediately diagonally above it.

				1						row 0
			1		1					row 1
		1		2		1				row 2
	1		3		3		1			row 3
1		4		6		4		1		row 4
				⋮						

What is the sum of the numbers in row 7?

(A) 64

(B) 65

(C) 66

(D) 67

(E) 128

SOLN The long way is to write three more rows in the table and add up the entries in row 7. The short way is to see that the sum of the entries in row n is 2^n. □

34. If p is a prime number and $p^3 + 5$ is also prime, then $p^5 + 7$ is

(A) prime.

(B) composite.

(C) either prime or composite.

(D) neither prime nor composite.

(E) Not enough information

SOLN The key is that the only even prime is 2. If p is odd, then p^3 is odd and $p^3 + 5$ is even, and therefore not prime. So the conditions can only be met for $p = 2$. And $2^3 + 5 = 13$ which is prime. Then $2^5 + 7 = 32$ which is composite. □

35. Simplify $\sqrt{1 - \cos^2 19°} - \sqrt{1 - \sin^2 98°}$.

(A) $\sin 19° - \cos 98°$

(B) $-\sin 19° - \cos 98°$

(C) $-\sin 19° + \cos 98°$

(D) $\sin 19° + \cos 98°$

(E) $-\sin 98° + \cos 19°$

SOLN $\sin^2\theta + \cos^2\theta = 1$ but watch the signs; $\sqrt{\ }$ refers to the positive root. $\sqrt{1 - \sin^2 98°} = -\cos 98°$ □

36. Find the median for the set of values.

3, 13, 4, 1, 4, 6, 7, 1, 5, 1

(A) 1

(B) 3

(C) 4

(D) 4.5

(E) 5

SOLN List them in ascending order and take the middle value or the average of the two middle values.

1, 1, 1, 3, 4, 4, 5, 6, 7, 13.

The two middle values are 4 and 4. □

37. Of the properties commutative, associative, and identity, which hold for the operator $\otimes$ given in the table?

$\otimes$	a	b	c
a	a	b	c
b	b	a	b
c	c	b	a

(A) commutative and identity only

(B) commutative and associative only

(C) identity and associative only

(D) commutative only

(E) all three

SOLN The identity is a. The table is symmetric across the main diagonal so the operator is commutative. But it is not associative because
$(b \otimes b) \otimes c = a \otimes c = c$ while
$b \otimes (b \otimes c) = b \otimes b = a$. □

38. Which statement about quadrilaterals is NOT correct?

(A) All rhombuses are parallelograms.

(B) All trapezoids are parallelograms.

(C) All squares are rhombuses.

(D) All squares are rectangles.

(E) All rectangles are parallelograms.

SOLN Trapezoids must have two parallel sides, but the other two don't have to be parallel; in a parallelogram both sets of opposite sides must be parallel. □

39. The repeating decimal $0.\overline{63} = 0.636363\ldots$ can be represented as a rational number p/q where p and q are relatively prime integers. What is $p+q$?

(A) 16

(B) 18

(C) 21

(D) 63

(E) 77

SOLN If $x = 0.\overline{63}$, then $100x = 63.\overline{63}$. Subtract

$$\begin{array}{rcl} 100x & = & 63.\overline{63} \\ x & = & 0.\overline{63} \\ \hline 99x & = & 63 \end{array}$$

$$x = \frac{63}{99} = \frac{7}{11} = \frac{p}{q}$$ □

40. What is the inverse of the matrix?

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

(A) $\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

(B) $\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

(C) $\frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

(D) $\frac{1}{3}\begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$

(E) $\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

SOLN If A is a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which is invertible, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ □