# Snow College Mathematics Contest

key

**April 4, 2023** **Senior Division: Grades 10-12** **Form: T**

Bubble in clearly the single best choice for each question you choose to answer.

1. Define $v_3(n) = m$ where $n = 3^m k$, where $m$ and $k$ are positive integers, and 3 is not a factor of $k$. $v_3(0) = \infty$. Then if $v_3(a) < v_3(b) < v_3(c)$, which is true?

   (A) $v_3(a+b+c) > v_3(a)$

   (B) $v_3(a+b+c) \geq v_3(c) + v_3(a)$

   (C) $v_3(a+b+c) > v_3(c)$

   (D) $v_3(a+b+c) \geq 2v_3(c)$

   $\boxed{\text{(E)}}$ $v_3(a+b+c) = v_3(a)$

   SOLN Let $a = 3^{m_1}k_1$, $b = 3^{m_2}k_2$, $c = 3^{m_3}k_3$ so that $v_3(a) = m_1 < v_3(b) = m_2 < v_3(c) = m_3$. Then $a+b+c = 3^{m_1}k_1 + 3^{m_2}k_2 + 3^{m_3}k_3 = 3^{m_1}(k_1 + 3^{m_2-m_1}k_2 + 3^{m_3-m_1}k_3)$ and then $v_3(a+b+c) = m_1 = v_3(a)$. In the case $c = 0$, $v_3(c) = \infty$ and the argument reduces to two terms. ▫

2. Suppose that a pyramid with a square base is inscribed inside of a sphere of radius 2. What is the area of the base of the pyramid if its base is exactly 1 unit below the center of the sphere? That is, $OE = 1$ and $OS = 2$.

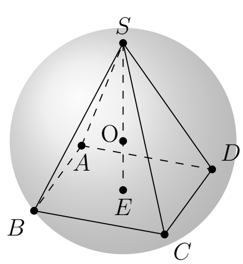
   (A) $3\sqrt{3}$

   (B) $4\sqrt{2}$

   $\boxed{\text{(C)}}$ 6

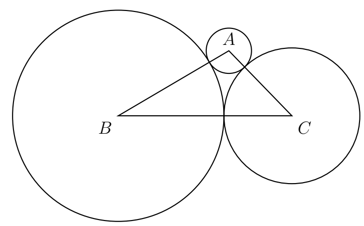
   (D) $2\sqrt{5}$

   (E) $\sqrt{3} + 4$




   SOLN Right triangle $OEC$ has a hypotenuse of 2 (the radius) and a leg ($OE$) of 1. So leg $EC = \sqrt{3}$. Right triangle $DEC$ has two legs of $\sqrt{3}$ so the hypotenuse is $CD = \sqrt{6}$. The base area is then $(\sqrt{6})^2 = 6$. ▫

3. Three mutually tangent circles have centers $A$, $B$, and $C$ and radii $a$, $b$, and $c$ respectively. The lengths of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are 17, 23, and 12 respectively. Find the lengths of the radii.




   (A) $a = 13, \quad b = 9, \quad c = 7$

   (B) $a = 5, \quad b = 12, \quad c = 6$

   (C) $a = 6, \quad b = 9, \quad c = 12$

   (D) $a = 4, \quad b = 8, \quad c = 12$

   $\boxed{\text{(E)}}$ $a = 3, \quad b = 14, \quad c = 9$

   SOLN We have $a + b = 17$, $b + c = 23$, and $c + a = 12$. Adding the equations gives $2(a+b+c) = 52 \Rightarrow a+b+c = 26$. Then subtract each of the original equations from this last equation. ▫

4. A room contains 3 people. Disregarding leap years, what is the probability that at least 2 of the 3 people have the same birthday?

(A) $\dfrac{365 \cdot 364 \cdot 363}{365^3}$

$\boxed{\text{(B)}}$ $1 - \dfrac{365 \cdot 364 \cdot 363}{365^3}$

(C) $\dbinom{365}{2}$

(D) $\dbinom{365}{2} \cdot 2!$

(E) $\dbinom{365}{2} \cdot 3!$

$\boxed{\mathcal{SOLN}}$ Let $A$ be the event that at least 2 individuals have the same birthday. Then $A^c$ is the event that no two people share the same birthday. Thus $P(A) = 1 - P(A^c) = 1 - \frac{365 \cdot 364 \cdot 363}{365^3}$. $\square$

5. Simplify $\log_{\sqrt{3}} 27$.

(A) $\sqrt[3]{2}$

(B) $3/2$

(C) $3$

$\boxed{\text{(D)}}$ $6$

(E) $7$

$\boxed{\mathcal{SOLN}}$ This is equivalent to $(\sqrt{3})^x = 27$ or $3^{x/2} = 3^3$. Thus $x/2 = 3 \Rightarrow x = 6$. $\square$

6. Suppose that there are 3 committees. Committees $A$, $B$, $C$ have 4, 2, and 1 people respectively. How many ways are there of forming a committee $D$ consisting of exactly 3 members from these committees such that no more than 2 people come from committee $A$ and at least one person comes from committee $B$?

(A) 21

(B) 22

(C) 23

(D) 24

$\boxed{\text{(E)}}$ 25

$\boxed{\mathcal{SOLN}}$

$$\binom{4}{0} \cdot \binom{2}{2} \cdot \binom{1}{1} + \binom{4}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} + \binom{4}{1} \cdot \binom{2}{2} \cdot \binom{1}{0} + \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{0} =$$

$$1 \cdot 1 \cdot 1 + 4 \cdot 2 \cdot 1 + 4 \cdot 1 \cdot 1 + 6 \cdot 2 \cdot 1$$ $\square$

7. Tommy likes to throw darts. He has found that 40% of the time he hits the outer circle (5 points), 30% of the time he hits the middle circle (10 points), 20% of the time he hits the inner circle (20 points), and 5% of the time the bullseye (50 points). 5% of the time he misses the the target (0 points). Each day he throws 3 darts. Which scenario is more likely in a day?

$\boxed{\text{(A)}}$ He hits at least one bullseye

(B) He hits the bullseye exactly twice

(C) He scores over 100

(D) He scores exactly 150

(E) He misses the target all three times

$\boxed{\mathcal{SOLN}}$ Choices B, C, and D are subsets of A. D and E are equally likely. $\square$

8. In a class of $p$ students, the average (arithmetic mean) of the test scores is 70. In another class of $n$ students, the average of the scores for the same test is 92. When the scores of the two classes are combined, the average of the test scores is 86. What is the value of $\frac{p}{n}$ ?

$\boxed{\text{(A)}}$ $\frac{3}{8}$

(B) 3

(C) 4

(D) 8

(E) Not enough information

$\boxed{\text{SOLN}}$ $86 - 70 = 16$ so the total from the mean of the first class is $16p$. $92-86 = 6$ so the total from the mean of the second class is $6n$. The deficit of points has to be equal to the surplus of points, so $16p = 6n$. □

9. Find the sum of all the positive factors (divisors) of 400.

(A) 561

(B) 759

(C) 903

$\boxed{\text{(D)}}$ 961

(E) 981

$\boxed{\text{SOLN}}$ Brute force works but is inelegant. The prime factorization of 400 is $2^4 \cdot 5^2$. Any divisor will be of the form $2^x \cdot 5^y$, where $x \in \{0, 1, 2, 3, 4\}$ and $y \in \{0, 1, 2\}$. This is summarized in the table.

| | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ |
|---|---|---|---|---|---|
| $5^0$ | 1 | 2 | 4 | 8 | 16 |
| $5^1$ | 5 | 10 | 20 | 40 | 80 |
| $5^2$ | 25 | 50 | 100 | 200 | 400 |

Note each row or column forms a geometric series. The factoring yields a total sum of $(1+2+4+8+16) \times (1+5+25) = 31 \times 31 = 961$. □

10. There are several kinds of averages or means. One of them is the geometric mean, which is used often in computing growth rates. The geometric mean of $n$ numbers is defined as $(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{(1/n)}$. Compute the geometric mean of 9, 15, and 25.

(A) $\frac{49}{3}$

$\boxed{\text{(B)}}$ 15

(C) $\frac{225}{49}$

(D) 16

(E) $\frac{79}{7}$

$\boxed{\text{SOLN}}$ $(9 \cdot 15 \cdot 25)^{\frac{1}{3}} = (3^3 \cdot 5^3)^{\frac{1}{3}} = 3 \cdot 5$ □

11. The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of equal size is their element-wise product $A \circ B \equiv [a_{ij}b_{ij}]$. This product differs from the usual matrix product $AB$. Which of the following is not a true property of the Hadamard product?

(A) $A \circ B = B \circ A$

(B) $A \circ (B + C) = A \circ B + A \circ C$

(C) The Hadamard identity matrix consists of all ones.

(D) The Hadamard inverse exists only if no entry is zero.

$\boxed{\text{(E)}}$ $A \circ B = BA$

$\boxed{\text{SOLN}}$ The Hadamard product (also called the Schur product) is commutative and distributive over addition because real number multiplication is. The matrix inverse exists only if each entry has a real multiplicative inverse. Thus no entry can be zero. The identity of real number multiplication is 1, so the identity matrix consists of all ones.

This product is unrelated to the usual matrix multiplication in which for $m \times n$ matrices $A$ and $B$, $AB$ or $BA$ is only defined if $m = n$. □

12. What is the probability that three rolled, fair, six-sided dice will all show different numbers?

$\boxed{\text{(A)}}$ $\frac{5}{9}$

(B) $\frac{5}{27}$

(C) $\frac{1}{3}$

(D) $\frac{2}{3}$

(E) $\frac{2}{9}$

SOLN The second has a probability of 5/6 of being different. The third has a probability of 4/6 of being different from the first two. $\frac{5}{6} \cdot \frac{4}{6} = \frac{20}{36}$ □

13. If $a_{k+2} - a_{k+1} - 2a_k = 0$ for $k = 0, 1, 2, \ldots$, find $a_6$ if $a_0 = 9$ and $a_1 = -12$.

$\boxed{\text{(A)}}$ $-54$

(B) $-32$

(C) $27$

(D) $54$

(E) $64$

SOLN The sequence can be solved as $a_{k+2} = a_{k+1} + 2a_k$. We have $a_2 = a_1 + 2a_0 = -12 + 2(9) = 6$. Then $a_3 = a_2 + 2a_1 = 6 + 2(-12) = -18$. Then $a_4 = a_3 + 2a_2 = -18 + 2(6) = -6$. Then $a_5 = a_4 + 2a_3 = -6 + 2(-18) = -42$. Then $a_6 = a_5 + 2a_4 = -42 + 2(-6) = -54$ □

14. In the following figure, circles $A, B, C$, and $D$ are all radius 1 and tangent to each other. What is the area of the shaded region?

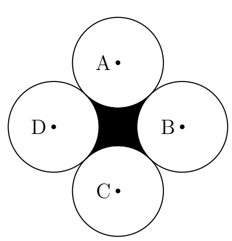
(A) $\frac{\pi}{4}$

(B) $1$

$\boxed{\text{(C)}}$ $4 - \pi$

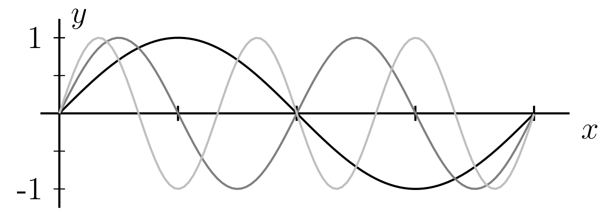
(D) $\pi$

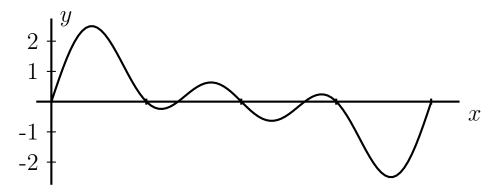
(E) $4$



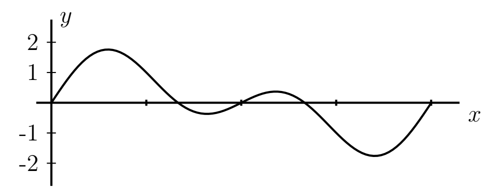

SOLN Connecting points $A, B, C$, and $D$ forms a square of area 4. The four quarter circles cut out of the square make up one circle with a radius of 1 and an area of $\pi$ square units. □

15. Below are graphs of $y = \sin x$, $y = \sin 2x$, and $y = \sin 3x$ on the interval $[0, 2\pi]$. Which of the following graphs shows $y = \sin x + \sin 2x + \sin 3x$?

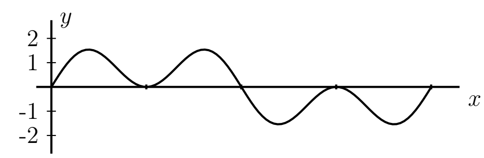


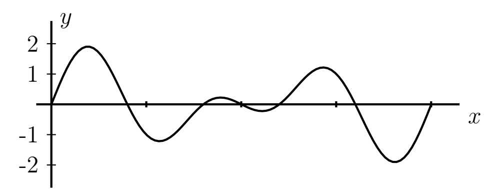

$\boxed{\text{(A)}}$



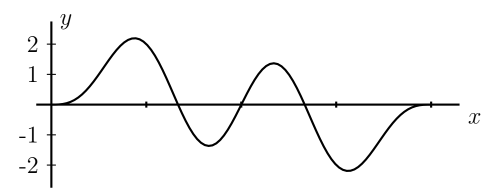

(B)




(C)



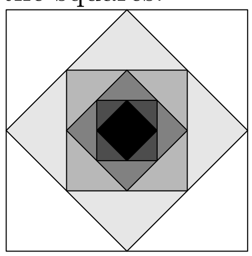

(D)




(E)

SOLN For any $x$ value, the $y$ value of the sum is the sum of the $y$ values of the individual functions. At the first and third tick marks the three functions have values near $-1, 0, 1$, the sum of which is 0; choices (A) and (C). About halfway between the $y$-axis and the first tick mark the values sum to more than 2. □

16. Only the first six of a sequence of squares are shown. The outermost square has an area of $8\,\text{cm}^2$. Each of the other squares is obtained by joining the midpoints of the sides of the previous square. Find the sum of the infinite series of the areas of all the squares.



(A) $12\,\text{cm}^2$

$\boxed{\text{(C)}}$ $16\,\text{cm}^2$

(B) $14\,\text{cm}^2$

(D) $20\,\text{cm}^2$

(E) $100\,\text{cm}^2$

$\boxed{\textit{SOLN}}$ The sum of the areas is a geometric series $8 + 4 + 2 + 1 + \cdots$. The series converges to $8/(1 - 1/2) = 16\,\text{cm}^2$. □

17. Line $y = mx + b$ is tangent to the circle $(x + 1)^2 + (y - 1)^2 = 25$ at $(3, 4)$. Find $m+b$.

(A) $\frac{5}{12}$

(B) $\frac{5}{2}$

(C) $\frac{7}{2}$

$\boxed{\text{(D)}}$ $\frac{20}{3}$

(E) $\frac{35}{4}$

$\boxed{\textit{SOLN}}$ Implicit differentiation for $y'(3, 4)$.
$2(x + 1) + 2(y - 1)y' = 0$
$(y - 1)y' = -(x + 1) \Rightarrow y' = \frac{-(x+1)}{y-1}$
$y'(3, 4) = \frac{-4}{3} = m$. Plug $(3, 4)$ and $m$ into $y = mx + b$ to find $b = 8$. Second solution: find the slope of the radial line to (3,4), then the tangent line must be perpendicular to that. (-4/3). □

18. Average angular velocity is defined as angular displacement divided by time. A year contains approximately $\pi \times 10^7$ seconds. What is the angular velocity of the earth as it travels around the sun?

(A) $10^{-7}$ rad/s

$\boxed{\text{(B)}}$ $2 \times 10^{-7}$ rad/s

(C) $10^7$ rad/s

(D) $360 \times 10^{-7}$ rad/s

(E) $180 \times 10^{-7}$ rad/s

$\boxed{\textit{SOLN}}$ Divide the angular displacement over a year, which is $2\pi$ rad, by the time. □

19. Suppose that ☺$(n)$ is a function of positive integers which returns the smallest prime number that is equal to or larger than $n$. Find ☺$(159)$ + ☺$(44)$.

(A) 203

$\boxed{\text{(B)}}$ 210

(C) 213

(D) 222

(E) 238

$\boxed{\textit{SOLN}}$ 159 is divisible by 3 and 161 is divisible by 7 so ☺$(159) = 163$ (verify by division by 3, 5, 7, and 11; $13^2 > 163$). ☺$(44) = 47$. □

20. Kim plays basketball for her school. Her free-throw shooting percentage for the season was 76% exactly before today. During tonight's game she makes all five free throws, bringing her percentage up to 80%. How many free throws has Kim made for the entire season (including tonight)?

- $\boxed{\text{(A)}}$ 24
- (B) 25
- (C) 26
- (D) 28
- (E) 30

SOLN Call her shots taken before tonight $x$. Her shots made can be represented as

$$0.76x + 5 = 0.80\,(x + 5)$$

where the left side is shots made before tonight + 5, and the right side is 80% of all shots made. Solving for $x$ gives $x = 25$; then the left side becomes $19 + 5 = 24$. □

21. Consider the following recursive function. If $n = 0$, then $f(0) = 1$. For $n$ an integer greater than zero, $f(n) = n \cdot f(n-1)$. What does this recursion compute?

- $\boxed{\text{(A)}}$ $n!$
- (B) $n^n$
- (C) $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$
- (D) $n \cdot (n-1) \cdot 1$
- (E) $n^{n-1}$

SOLN The recursion is the factorial: $f(n) = n! = n(n-1)(n-2)\ldots 1$. □

22. Lucy, Ricky, Ethel, and Fred all go to a party where the following pie options are available: apple, pumpkin, chocolate cream, and pecan. If attendees are allowed to eat at most one piece of each pie flavor, use the following clues to determine which types of pie were eaten by each of the four friends.

- Ricky is allergic to chocolate, and he detests pumpkin pie.
- Ricky and Ethel were the only ones to eat apple pie.
- Lucy ate only one piece of pie while the other three each ate two pieces.
- Lucy and Ethel each ate a piece of chocolate cream pie.
- Only one of the four friends ate a piece of pumpkin pie.
- Between the four friends, no more than two pieces of any one flavor were eaten.
- The four friends ate a combined total of seven pieces of pie.

Which of the friends ate pecan pie?

- (A) Ricky, Ethel
- (B) Lucy, Fred
- (C) Ricky, Lucy
- $\boxed{\text{(D)}}$ Ricky, Fred
- (E) Lucy, Ethel

SOLN Ricky ate two pieces, one of which was apple, and he did not eat pumpkin or chocolate. So Ricky must have eaten a piece of pecan. Fred did not eat apple or chocolate as others ate the two pieces for those flavors. Thus Fred's two pieces must have been pumpkin and pecan.

| | App | Pum | CC | Pec |
|---|---|---|---|---|
| Lucy | X | X | O | X |
| Ricky | O | X | X | O |
| Ethel | O | X | O | X |
| Fred | X | O | X | O |

□

23. Consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$.

$$\begin{array}{ll} q^1 \div p \text{ has remainder } a_1 \\ q^2 \div p \text{ has remainder } a_2 \\ q^3 \div p \text{ has remainder } a_3 \\ \vdots & \vdots \end{array}$$

No matter the choice of integers $p$ and $q$, this sequence $a_n$ *always has a repeating pattern*! Use this idea to determine which of the following is true of $2^{241} - 2$.

(A) It must be a multiple of 3 but not 5.

(B) It must be a multiple of 5 but not 3.

$\boxed{\text{(C)}}$ It must be a multiple of 3 and 5.

(D) It must be a multiple of neither 3 nor 5.

(E) Not enough information

$\boxed{\mathcal{SOLN}}$ It must be a multiple of 3 and 5.

• Let $q = 2$. Then if $p = 3$, the sequence is $2, 1, 2, 1, 2, 1, \ldots$. Since 241 is odd, $2^{241} \div 3$ has remainder 2. Therefore, $2^{241} - 2$ is an exact multiple of 3.

• Let $q = 2$. Then if $p = 5$, the sequence is $2, 4, 3, 1, 2, 4, 3, 1, 2, \ldots$ with a repeating block of size 4. Since 241 is one more than a multiple of 4, $2^{241} \div 5$ has remainder 2. Therefore, $2^{241} - 2$ is an exact multiple of 5. □

24. Since $2\pi$ shows up in so many physics formulas, it has been proposed that it be replaced with a single symbol $\tau$ to simplify.
Evaluate Euler's remarkable formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

at $\theta = \tau$ rad.

(A) $-1$

(B) $0$

$\boxed{\text{(C)}}$ $1$

(D) $\pi$

(E) $\tau$

$\boxed{\mathcal{SOLN}}$ $\tau$ rad is one full turn, so
$e^{i\tau} = e^0 = \cos(0) + i\sin(0) = 1 + 0$.
`https://tauday.com` □

25. An *algebraic number* is any number that is a root of a non-zero polynomial with rational coefficients. The golden ratio $\phi = \frac{1}{2}(1 + \sqrt{5})$ is an irrational algebraic number. Which polynomial is it a root of?

$\boxed{\text{(A)}}$ $x^2 - x - 1$

(B) $x + \sqrt{5}$

(C) $x^3 + x^2 + x + 1$

(D) $5x^2 + 2x + 1$

(E) $5x^3 - 2x^2 - x - 1$

$\boxed{\mathcal{SOLN}}$ $\phi^2 - \phi - 1 =$
$\frac{1}{4}(1 + 2\sqrt{5} + 5) - \frac{1}{2}(1 + \sqrt{5}) - 1 = 0$ □

26. A *transcendental number* is any number that is **not** a root of a non-zero polynomial with rational coefficients. The imaginary $\sqrt{-1} = \mathrm{i}$ is not transcendental (it is the root of $x^2+1$), but $\mathrm{i}^{\mathrm{i}}$ is. Express $\mathrm{i}^{\mathrm{i}}$ in an alternate form using Euler's remarkable formula $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and $\tau = 2\pi$.

(A) $\sin^2(\tau) + \cos^2(\tau)$

(B) $\sin(\mathrm{e}^{\tau})$

(C) $\mathrm{e}^{\sin(\tau)}$

(D) $\cos(\tau/2)$

$\boxed{\text{(E)}}$ $\mathrm{e}^{-\tau/4}$

$\boxed{\mathcal{SOLN}}$ $\mathrm{e}^{\mathrm{i}(\tau/4)} = \cos(\tau/4) + \mathrm{i}\sin(\tau/4) = \mathrm{i}$
Raise both sides to the i power.
$\mathrm{e}^{\mathrm{i}(\tau/4)\mathrm{i}} = \mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}^2(\tau/4)} \approx 0.207879576$
`https://tinyurl.com/3kg3g7h` □

27. What is the least number of prime factors (not necessarily different) that 700 must be multiplied by so that the product is a perfect cube?

(A) 1

(B) 2

(C) 3

$\boxed{\text{(D)}}$ 4

(E) 5

$\boxed{\mathcal{SOLN}}$ For a number to be a perfect cube each prime in the prime factorization must be cubed. $700 = 2^2 \cdot 5^2 \cdot 7$ so we need one more factor of 2, one more factor of 5, and two more of 7. □

28. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find $B$.

$\boxed{\text{(A)}}$ $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$

(E) $\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$

$\boxed{\mathcal{SOLN}}$ Since $AB = I$ then $B = A^{-1}$. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ □

29. What is the area of the largest rectangle that can be inscribed in a closed semicircle of radius 4?

(A) 32

$\boxed{\text{(B)}}$ 16

(C) $8\sqrt{2}$

(D) $16\sqrt{2}$

(E) $12\sqrt{3}$

$\boxed{\mathcal{SOLN}}$ Easy: Make a complete circle and notice the largest area rectangle inscribed in the circle is a square of side length $4\sqrt{2}$. This square has area 32 so the semicircle rectangle will have area 16.
Hard: Model the semicircle with $x^2 + y^2 = 16$. Then the area of the rectangle will be $2xy$. Use calculus to optimize the area of the rectangle. □

30. If I multiply the number of math contests I have taken in my life by 6 and then add 5, the resulting number cannot be divisible by which number?

(A) 5

(B) 7

$\boxed{\text{(C)}}$ 9

(D) 11

(E) 13

$\boxed{\text{SOLN}}$ Multiplying any whole number by 6 results in a product divisible by 3; after adding 5, the sum can no longer be divisible by 3 and therefore not by 9. □

31. If you place a cake of soap on a pan of a scale and $\frac{3}{4}$ cake of soap and a $\frac{3}{4}$-kg weight on the other, the pans balance. How much does a cake of soap weigh?

$\boxed{\text{(A)}}$ 3 kg

(B) 1 kg

(C) $\frac{3}{4}$ kg

(D) $\frac{1}{2}$ kg

(E) $\frac{1}{4}$ kg

$\boxed{\text{SOLN}}$ Since $\frac{1}{4}$ cake weighs $\frac{3}{4}$ kg, an entire cake weighs 3 kg.

$$x = \frac{3}{4}x + \frac{3}{4} \implies x = 3$$ □

32. Which is the correct description of the graph of $(x+y)^2 = x^2 + xy + y^2$ ?

$\boxed{\text{(A)}}$ Two intersecting lines

(B) The empty set

(C) A single point

(D) Two parallel lines

(E) A circle

$\boxed{\text{SOLN}}$ Expanding $(x+y)^2$ gives $x^2+2xy+y^2$. Then $x^2+2xy+y^2 = x^2+xy+y^2$ leaves us with $xy = 0$. The solutions of this are $x = 0$ or $y = 0$ which are perpendicular intersection lines. □

33. Which of the following is equivalent to the value of $\tan 10°$?

$\boxed{\text{(A)}}$ $\cot 80°$

(B) $\tan 80°$

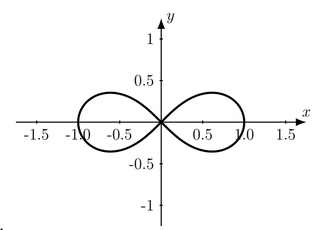
(C) $\sec 10°$

(D) $\csc 80°$

(E) $\tan 55°$

$\boxed{\text{SOLN}}$ The cofunctions of complementary angles are equal. □

34. The polar graph shown below is the lemniscate with equation $r^2 = \cos 2\theta$. Which Cartesian equation of conic sections is equivalent to the inverse curve (reciprocal) of the lemniscate?




(A) $y = x^2$

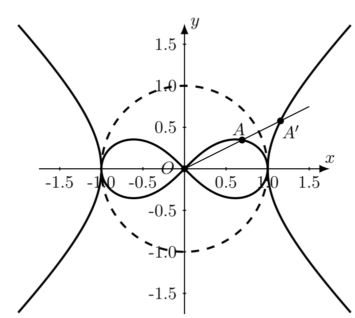
(B) $x^2 + y^2 = 1$

$\boxed{\text{(C)}}$ $x^2 - y^2 = 1$

(D) $y^2 - x^2 = 1$

(E) $x^2 + y^2/2 = 1$

$\boxed{\text{SOLN}}$ The lemniscate's reciprocal is $\frac{1}{r^2} = \cos 2\theta$ or $r^2 = \frac{1}{\cos 2\theta}$. Multiplying and applying the cosine double angle identity gives $r^2 \cos 2\theta = r^2 \cos^2\theta - r^2 \sin^2\theta = x^2 - y^2 = 1$.




This is called inversion through a circle. For a circle of radius $r$ and center $O$, the distance $OA$ and the inverted distance $OA'$ are related by $OA \cdot OA' = r^2$. □

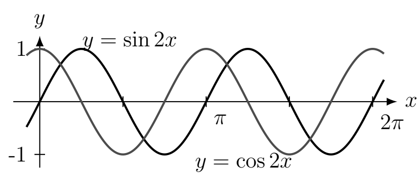
35. How many distinguishable ways are there to rearrange the letters in the word MINIMUM (example: "MMIINMU").

(A) 70
(B) 120
(C) 240
$\boxed{\text{(D)}}$ 420
(E) 840

SOLN A complete listing works but is very time consuming. There are 7 letters in the word so $7! = 5040$ ways to write the letters. However, since there are 3 Ms and 2 Is, we have to take out the double-counting by dividing by 3! and 2! Respectively. $5040/(6 \cdot 2) = 420$. □

36. How many solutions are there to $\cos(2x) = \sin(2x)$ on the interval $[0, 2\pi]$?

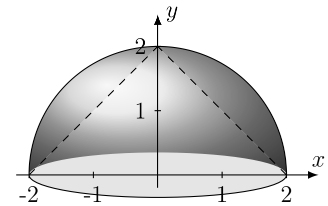
(A) 0
(B) 1
(C) 2
$\boxed{\text{(D)}}$ 4
(E) 6




SOLN It can be solved using trig identities but simply graphing the two functions is quicker. Alternatively, this is like asking the number of solutions of $\cos(x) = \sin(x)$ on $[0, 4\pi]$, where $x = \pi/4,\ 5\pi/4,\ 9\pi/4,\ 13\pi/4$.
The 4 solutions to original problem are $x = \pi/8,\ 5\pi/8,\ 9\pi/8,\ 13\pi/8$. □

37. What is the volume of the region formed by revolving the area in the first quadrant between $y = \sqrt{4 - x^2}$ and $y = 2 - x$ around the $y$-axis?

$\boxed{\text{(A)}}$ $8\pi/3$
(B) $2\pi$
(C) $3\pi$
(D) $5\pi/2$
(E) $17\pi/6$

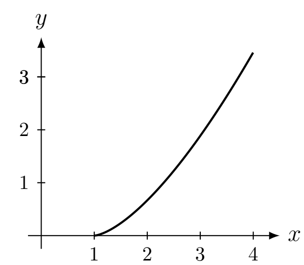



SOLN Can be solved using integrals but 3-D geometry is easier. $V_{\text{sphere}} = \frac{4\pi}{3}r^3$ and $V_{\text{cone}} = \frac{\pi}{3}r^2 h$. A half sphere of radius 2 has volume $\frac{16\pi}{3}$ while the cone will have volume $\frac{8\pi}{3}$. This is a difference of $\frac{8\pi}{3}$. □

38. What is the length of the curve?

$$y = \tfrac{2}{3}(x-1)^{\frac{3}{2}}, \qquad 1 \le x \le 4$$

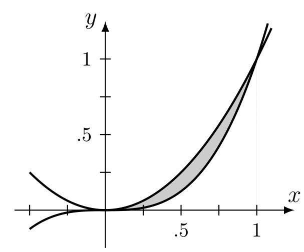
(A) $\frac{12\sqrt{3}}{5}$
(B) 9
$\boxed{\text{(C)}}$ $\frac{14}{3}$
(D) 12
(E) $2\sqrt{3}$




SOLN $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + (x-1)}\, dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}[8 - 1] = \frac{14}{3}$ □

39. What is the area between $y = x^2$ and $y = x^3$ between $x = 0$ and $x = 1$?

$\boxed{\text{(A)}}$ $\frac{1}{12}$
(B) $\frac{\pi}{4}$
(C) $\frac{1}{8}$
(D) $\frac{2}{5}$
(E) $\frac{4}{9}$




SOLN $\int_0^1 (x^2 - x^3)dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ □

40. What is the maximum height that a 20 ft wide truck can have in order to barely pass under a parabolic arch with a height of 25 ft and a base width of 50 ft?

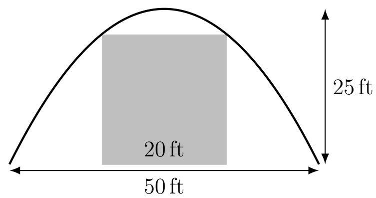
(A) 16 ft

(B) 20 ft

$\boxed{\text{(C)}}$ 21 ft

(D) 22 ft

(E) 24 ft




$\boxed{\mathcal{SOLN}}$ The parabola is inverted and shifted up 25 ft; equation is $y = -ax^2 + 25$. Substituting the point $(25, 0)$ determines that $a = 1/25$. The truck will pass most easily when it is centered along the axis of symmetry. Substituting $x = 10$ gives the maximum height possible. $\square$