Snow College Mathematics Contest

April 5, 2022 **Senior Division: Grades 10-12** **Form: $\boxed{\text{T}}$**

Please read all instructions on this page very carefully.

1. Leave this booklet closed until you are instructed to begin.
2. Bubble your test version (Form T) on the upper left side of the bubble sheet. Make sure your bubble sheet has your name already printed (if not, return to the registration table).
3. This is a two hour examination consisting of 40 multiple choice problems. Bubble in the single best answer to each question you choose to answer clearly on the answer sheet. Completely erase any answers you wish to change.
4. Avoid random guessing as there is a penalty for wrong answers. There is no penalty for leaving a question blank. The formula for scoring the test is $\text{Score} = 40 + 4R - W$ where R and W denote the number right and wrong respectively. The possible scores range from 0 to 200.
5. Ties will be broken by the first discrepancy in the following five problems *in order*: 1, 2, 4, 9, 10. In the event of no discrepancies in those problems, the tie will be broken by a coin toss.
6. No calculators are allowed. Diagrams are not necessarily drawn to scale.
7. Do not talk or disrupt other test takers during the exam. Cell phones must be OFF (not just on silent or vibrate, but OFF). No ear buds are allowed.
8. Please raise your hand if you need scratch paper or a new pencil; a proctor will assist you.
9. The proctors have been advised to not answer questions pertaining to the exam.
10. If you have time we recommend you recheck your answers. If you finish early you may quietly turn your answer sheet in and leave. After the two hour time limit is up the proctors will call for all answer sheets; hand them in promptly.

After the test:

1. You may keep this test booklet and the pencils.
2. If you RSVP'd to spend time with one of our science departments for lunch, please meet them in the science center (GRSC) or library auditorium; otherwise lunch may be purchased at the Snow College Cafeteria or downtown. In any event, you should plan to be back at the LDS Institute by 1:00 p.m. for the scores and presentation of the awards.
3. Top individual and team scorers in each classification will receive prizes.
4. Thanks for coming; we hope you had fun and learned some math. Your instructors will be happy to work the problems for you. They will also be given copies of your answer sheets.

Snow College Mathematics Contest

$\boxed{\text{key}}$

April 5, 2022 **Senior Division: Grades 10-12** **Form: $\boxed{\text{T}}$**

Bubble in the single best choice for each question you choose to answer.

1. Evaluate the integral. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

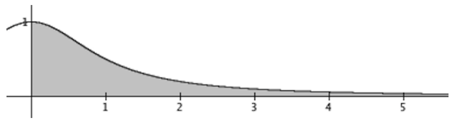
   $\boxed{\text{(A)}}$ $\frac{\pi}{2}$

   (B) $\frac{\pi}{4}$

   (C) $\frac{\pi}{3}$

   (D) $\sqrt{2}$

   (E) $\sqrt{3}$



   $\boxed{\mathcal{SOLN}}$ $\displaystyle\lim_{b\to\infty}\Big[\arctan x\Big]_0^b = \frac{\pi}{2} - 0$ □

2. Given three distinct two-digit numbers, what is the probability that two of them add up to the third one?

   (A) $\frac{1600}{90\cdot 89\cdot 88}$

   $\boxed{\text{(B)}}$ $\frac{1600\cdot 6}{90\cdot 89\cdot 88}$

   (C) $\frac{6400}{90\cdot 89\cdot 88}$

   (D) $\frac{6400\cdot 6}{90\cdot 89\cdot 88}$

   (E) $\frac{1}{10}$

   $\boxed{\mathcal{SOLN}}$ There are ${}_{90}C_3 = \frac{90\cdot 89\cdot 88}{3\cdot 2\cdot 1}$ possible trios. If 10 is the smallest of the three, the numbers 21 through 99 are all possible sums, giving 79 sums. If 11 is the smallest, the numbers 23 through 99 are all possible sums, giving another 77 sums, etc. There are $79 + 77 + \cdots + 1 = 40^2 = 1600$ total possible sums, giving a probability of $\frac{1600\cdot 6}{90\cdot 89\cdot 88}$. □

3. Solve for x (with $a > 0$): $2\log_b x = 2\log_b(1-a) + 2\log_b(1+a) - \log_b(\frac{1}{a} - a)^2$

   $\boxed{\text{(A)}}$ $x = a$

   (B) $x = \frac{1}{a}$

   (C) $x = a^2$

   (D) $x = 1 - a$

   (E) $x = \log_b a$

   $\boxed{\mathcal{SOLN}}$

$$2\log_b(1-a)+2\log_b(1+a)-\log_b(\tfrac{1}{a}-a)^2$$
$$= 2\log_b(1-a)+2\log_b(1+a)-2\log_b(\tfrac{1}{a}-a)$$
$$= 2\log_b(1-a)+2\log_b(1+a)-2\log_b(\tfrac{1}{a}(1-a^2))$$
$$= 2\log_b\left(\frac{1-a^2}{\frac{1}{a}(1-a^2)}\right)$$
$$= 2\log_b(a)$$ □

4. Towns A, B, and C are at the corners of a triangle with equal sides. A car travels at constant speed from A to B at 30 mph, from B to C at 40 mph, and from C back to A at 60 mph. What is the average speed for the round trip?

   $\boxed{\text{(A)}}$ 40 mph
   (B) 43 mph
   (C) 45 mph
   (D) 48 mph
   (E) 50 mph

   $\boxed{\mathcal{SOLN}}$ The answer is the same for any size triangle, but let's assume a specific case for ease of illustration: $s = 120$ mi, so the total trip is 360 mi. The first leg takes $(120\,\text{mi})/(30\,\text{mi/h}) = 4$ h; likewise, the second leg takes 3 h, and the last leg takes 2 h, for a total of 9 h. The average speed is $360\,\text{mi}/9\,\text{h} = 40\,\text{mi/h}$.

   Marilyn vos Savant in <u>Parade</u>, Dec. 17, 2017. □

5. How many ways are there of permuting the letters in this arrow diagram such that the arrow relationships $A \leftrightarrow B$, $C \leftrightarrow D$, and $E \to F \to G \to E$ are preserved?

$$\begin{tikzcd} A \arrow[d, leftrightarrow] & C \arrow[d, leftrightarrow] & E \arrow[r] & F \arrow[dl] \\ B & D & G \arrow[u] & \end{tikzcd}$$

   One acceptable permutation follows.

$$\begin{tikzcd} C \arrow[d, leftrightarrow] & B \arrow[d, leftrightarrow] & E \arrow[r] & F \arrow[dl] \\ D & A & G \arrow[u] & \end{tikzcd}$$

   (A) 6
   (B) 11
   $\boxed{\text{(C)}}$ 24
   (D) 144
   (E) 504

   $\boxed{\mathcal{SOLN}}$ There are 8 ways of switching around A, B, C, D and 3 of switching E, F, G. $24 = 3 \cdot 8$. □

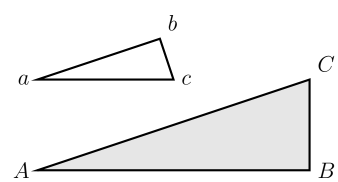
6. Em and Jo are each at home and want to meet up at the SnoSpud which is 12 mi from Em's house and 15 mi from Jo's house (as the crow flies). If the SnoSpud makes a 60° angle with their houses, estimate how far apart they live (as the crow flies).

   (A) 8.9 mi
   $\boxed{\text{(B)}}$ 13.7 mi
   (C) 17.6 mi
   (D) 19.2 mi
   (E) 27.0 mi

   $\boxed{\mathcal{SOLN}}$ Law of cosines: $d^2 = 12^2 + 15^2 - 2(12)(15)\cos 60°$ $\quad d = \sqrt{d^2} = \sqrt{189}$ □

7. Zoe is holding a full container while walking. She begins to go up a ramp with slope m. What slope relative to the ramp must the container follow if she wants to keep the container's path parallel to the ground in front of the ramp?

   (A) $-\frac{1}{m}$
   $\boxed{\text{(B)}}$ $-m$
   (C) $\frac{1}{m}$
   (D) m^2
   (E) $-\frac{1}{m^2}$



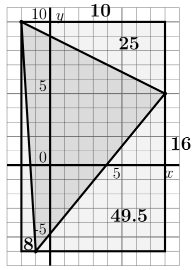
   $\boxed{\mathcal{SOLN}}$ AC is the ramp. ac is the path of the container. AC and ab are parallel; so are AB and ac by hypothesis. Similar triangles means $|\text{bc}/\text{ab}| = |\text{BC}/\text{AB}| = m$. For Zoe, however, the container *drops* in the vertical dimension, so we take bc to be negative. $\therefore$ $\text{bc}/\text{ab} = -m$. □

8. Find k so that the sum of the **squares** of the roots of $x^2 + 2x + k = 0$ is equal to 10.

   $\boxed{\text{(A)}}$ -3
   (B) -2
   (C) -1
   (D) 0
   (E) 1

   $\boxed{\mathcal{SOLN}}$ $x_1 + x_2 = -2 \Rightarrow (x_1 + x_2)^2 = 4$
   $x_1^2 + x_2^2 + 2x_1x_2 = 10 + 2k = 4$ □

9. Find the area (in square units) of the triangle that is created by connecting the three points $(-2, 10)$, $(-1, -6)$, and $(8, 5)$.



(A) 45.5

(B) 50.5

$\boxed{\text{(C)}}$ 77.5

(D) 80.5

(E) 160.5

SOLN

$$A = \left(\frac{1}{2}\right) \begin{vmatrix} -2 & 10 & 1 \\ -1 & -6 & 1 \\ 8 & 5 & 1 \end{vmatrix} = 77.5$$

Or, $A = 160 - 25 - 8 - 49.5 = 77.5$ □

10. How many ways are there to rearrange the letters of "math is fun" such that the word "math" is contained in every rearrangement? Ignore spaces.

(A) $9!$

$\boxed{\text{(B)}}$ $6!$

(C) $\binom{9}{4}$

(D) $6! \cdot 4!$

(E) $\binom{9}{4} \cdot 4!$

SOLN Since "math" is to appear in each permutation, we can think of "math" as being a single unit. There are 5 letters in "is fun"; including "math" as a single unit gives us 6 things to rearrange. Therefore there are $6!$ ways to rearrange the letters in "math is fun". □

11. If cars hold 5 passengers and charge \$29 a trip to the airport and vans hold 7 passengers and charge \$41, find the minimum cost to transport 49 people to the airport.

(A) \$290

$\boxed{\text{(B)}}$ \$285

(C) \$287

(D) \$280

(E) \$282

SOLN $\$29/5 < \$41/7$ so we want to rely on full cars. Full vans are better than a car with an empty seat.

vehicles	passengers	cost
7 cars	35	\$203
2 vans	14	\$82
total	49	\$285

□

12. The matrix $A = \begin{bmatrix} a & 8 \\ -3 & b \end{bmatrix}$ is its own inverse, i.e., $AA^{-1} = A^{-1}A = AA = I$. Find $|a - b|$.

(A) 4

(B) 6

(C) 8

$\boxed{\text{(D)}}$ 10

(E) 12

SOLN $\begin{bmatrix} a & 8 \\ -3 & b \end{bmatrix} \begin{bmatrix} a & 8 \\ -3 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^2 - 24 & 8a + 8b \\ -3a - 3b & -24 + b^2 \end{bmatrix}$

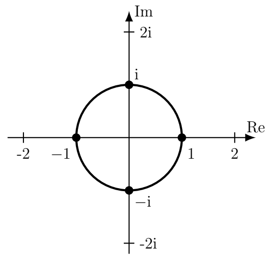
$a^2 - 24 = 1 \Rightarrow a^2 = 25 \Rightarrow a = \pm 5$

$-24 + b^2 = 1 \Rightarrow b^2 = 25 \Rightarrow b = \pm 5$

$8a + 8b = 0 \Rightarrow a = -b$

Alt solution: If $A = A^{-1}$, $\det(A) = \det(A^{-1}) = \pm 1 \Rightarrow ab - (-24) = \pm 1 \Rightarrow ab = -24 \pm 1$. -23 doesn't work, so $ab = -25$ □

13. Find the sum of the four fourth roots of 1.



$\boxed{\text{(A)}}$ 0

(B) $\frac{1}{2}$

(C) 1

(D) 2

(E) 4

$\boxed{\mathcal{SOLN}}$ The fundamental theorem of algebra says there are n complex n^{th} roots of 1. Since $e^{2\pi k i} = 1$ for k an integer, $e^{\frac{2\pi k i}{n}} = 1^{\frac{1}{n}}$ for any integer n. These n roots are equally spaced around the unit circle and they always sum to zero. For $n = 4$ we have $1^{1/4} = \{e^0, e^{\frac{\pi}{2}i}, e^{\pi i}, e^{\frac{3\pi}{2}i}\}$ $= \{1, i, -1, -i\}$. □

14. A palindrome is a number that reads the same forward and backward, such as 2552. How many palindromes are there between 1000 and 9999 (including 9999)?

$\boxed{\text{(A)}}$ 90

(B) 100

(C) 121

(D) 212

(E) 3024

$\boxed{\mathcal{SOLN}}$ There are 9 choices for the first digit and 10 independent choices for the second digit. Once these are picked, the third and fourth digits are fixed.
$9 \times 10 \times 1 \times 1 = 90$ □

15. Computers use binary (base-two system). What is the value of the following binary number in our base-ten system? 1001011

(A) 14

$\boxed{\text{(B)}}$ 75

(C) 103

(D) 125

(E) 150

$\boxed{\mathcal{SOLN}}$ $(1 \times 64) + (1 \times 8) + (1 \times 2) + (1 \times 1)$ □

16. Many are familiar with the arithmetic mean where you add up all the values and divide by the number of values n. There is another mean called the *harmonic mean*:

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

Find the harmonic mean of 10, 12, and 15.

(A) $\frac{21}{1}$

(B) 11

$\boxed{\text{(C)}}$ 12

(D) $\frac{37}{3}$

(E) $\frac{43}{3}$

$\boxed{\mathcal{SOLN}}$
$$\frac{3}{\frac{1}{10} + \frac{1}{12} + \frac{1}{15}} = \frac{3}{\frac{15}{60}} = \frac{60}{5}$$

Harmonic mean is the reciprocal of the arithmetic mean of the reciprocals. □

17. Jaden has an unusual analog clock. The hour hand works like normal, but the minute hand travels backward (counter-clockwise) and completes one rotation in 30 minutes instead of 60 minutes (the minute hand points to the 12 on the hour and on the half-hour). How many times will the two hands be aligned in the four hours between 11:30 and 3:30?

(A) 7

(B) 8

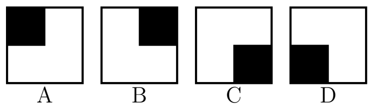
$\boxed{\text{(C)}}$ 9

(D) 10

(E) 11

$\boxed{\mathcal{SOLN}}$ The minute hand would typically pass the hour hand twice during each hour. But we will actually get three during the first hour: once shortly after 11:30, at 12:00, and shortly before 12:30. The next 3 hours will yield 6 more alignments for a total of 9. □

18. Plickers® is a quizzing method that uses a grid of pixels to convey information. A student shows the instructor a card which is read by a mobile device. Depending on the orientation of the pattern, one of four choices (A, B, C, D) is read. For example, a card using a 2×2 set of pixels could look like the following card in four orientations:



Each card must have a unique pattern for all four orientations. How many unique cards could be created using a 2×2 grid?

(A) 1

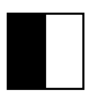
(B) 2

$\boxed{\text{(C)}}$ 3

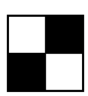
(D) 4

(E) 5

$\boxed{\mathcal{SOLN}}$ Besides the one shown, you could also create one using the inverse shading. A final card could be created with the following pattern:



Note that this next pattern would not work, as it would not create four unique patterns upon rotation:



Thus, the answer is 3. □

19. Suppose that $a_n = \frac{1}{4}(a_{n-1}^2 + 3)$. Then if $a_0 = 0$, compute $\lim_{n\to\infty} a_n$.

(A) ∞

(B) 0

$\boxed{\text{(C)}}$ 1

(D) 2

(E) 4

$\boxed{\mathcal{SOLN}}$ First notice that if a limit L does exist, then $L = \frac{1}{4}(L^2 + 3)$ so that $L^2 - 4L + 3 = 0$ or so $(L-1)(L-3) = 0$. Therefore, $L = 1$ or $L = 3$. Notice that if $|a_{n-1}| < 1$, then $|a_n| < 1$. □

20. An additive partition of 6 is an ordered sum such as $3 + 2 + 1 = 6$ such that the addends appear in nonincreasing order. How many additive partitions of 6 are there? For example, the additive partitions of 3 are 3, $2 + 1$, and $1 + 1 + 1$.

(A) 7

(B) 8

(C) 9

(D) 10

$\boxed{\text{(E)}}$ 11

6					
5	1				
4	2				
3	3				
4	1	1			
3	2	1			
2	2	2			
3	1	1	1		
2	2	1	1		
2	1	1	1	1	
1	1	1	1	1	1

$\boxed{\mathcal{SOLN}}$ List them; count them by hand. □

21. Snail racers (each 2 in long) are lined up with their tails touching the starting line and their bodies over (past) it. The race is 20 in from start to finish line and is over when any part of the snail crosses the finish line. The nose of one snail, Sluggard, moves 10 in in 2 min, then 5 in in the next 2 min, and so forth in an exponentially slowing way. Which value is *closest* to how long it will take Sluggard to finish the race?

   (A) Sluggard never finishes the race
   (B) 3 min
   (C) 4 min
   (D) 5 min
   $\boxed{\text{(E)}}$ 6 min

   SOLN Sluggard will finish after going 18 in. Sluggard totals 10 in, then 15 in, then 17.5 in, then 18.75 in. Sluggard does finish the race. □

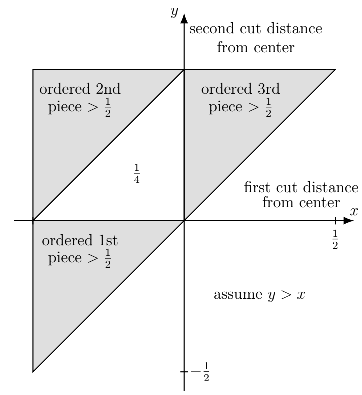
22. A car averages 26.9 mpg. On a sunny day that the car is driven, the car averages 29 mpg. On a stormy day that the car is driven the car averages 22 mpg. What percent of the days is the car driven on a stormy day?

   $\boxed{\text{(A)}}$ 30%
   (B) 40%
   (C) 50%
   (D) 60%
   (E) 70%

   SOLN Let x be the proportion of times it is sunny. Let $1-x$ be the proportion of times the car is driven when it is stormy. $29x + 22(1-x) = 26.9 \Rightarrow 29x + 22 - 22x = 26.9 \Rightarrow 7x = 4.9 \Rightarrow x = 0.7$ □

23. A yardstick is cut at 2 random places. What is the probability the sum of the two smaller pieces is longer than the largest piece?

   $\boxed{\text{(A)}}$ $\frac{1}{4}$
   (B) $\frac{1}{3}$
   (C) $\frac{1}{2}$
   (D) $\frac{2}{3}$
   (E) $\frac{3}{4}$



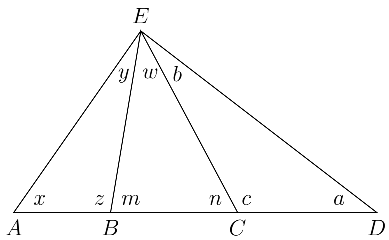
   SOLN In the graph, we require $y > x$ so that each coordinate pair (x, y) uniquely represents two cuts in the yard stick. If the positions of x and y are on the same side of the center (same sign), then one piece will have length $> \frac{1}{2}$. We omit this possibility by shading. We also omit the possibility that $y - x > \frac{1}{2}$ (which only happens when the cuts are on different sides of the center). The desired result (not shaded) is $\frac{1}{4}$ of the region of possible cuts. □

24. Free-fall acceleration on planet RedOrbo is $-4\,\mathrm{m/s^2}$. An explosion at ground level sends a rock flying vertically at an initial velocity of 12 m/s. How many seconds pass before the rock hits ground level? Ignore air resistance.

   (A) 3
   (B) 4
   $\boxed{\text{(C)}}$ 6
   (D) 8
   (E) 12

   SOLN $a(t) = -4 \Rightarrow v(t) = -4t + 12 \Rightarrow s(t) = -2t^2 + 12t$.
   For $s(t) = 0$ we get $t = 0, 6$. □

25. In a general triangle ADE, lines EB and EC are drawn. Which of the following angle relationships is true?



(A) $x + z = a + b$

(B) $y + z = a + b$

(C) $m + x = w + n$

(D) $x + z + n = w + c + m$

$\boxed{\text{(E)}}$ $x + y + n = a + b + m$

SOLN Triangle AEC: $x + n + y + w = 180°$
Triangle BED: $m + a + w + b = 180°$
Equate the left sides and subtract w. □

26. The graph shown is of $f'(x)$. If $f(2) = 5$, use the graph to find $f(8)$.

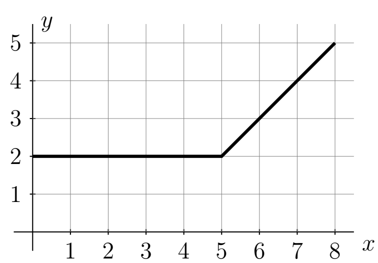
(A) 20.5

$\boxed{\text{(B)}}$ 21.5

(C) 22.5

(D) 23

(E) 25



SOLN We can geometrically find the area under the curve between 2 and 8 ($= 16.5$). Then, using the fact that $\int_2^8 f'(x)\,\mathrm{d}x = f(8) - f(2)$ we can add $f(2) = 5$ to the area to find that $f(8) = 16.5 + 5$. □

27. The points $(3, 3), (6, 6)$, and $(10, -4)$ are on a circle. What is the area of the circle?

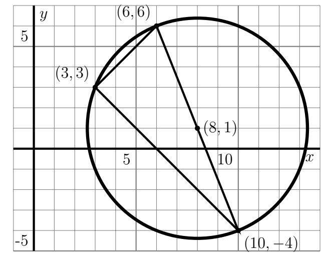
(A) $\sqrt{25}\pi$

(B) $\sqrt{29}\pi$

(C) 25π

(D) 27π

$\boxed{\text{(E)}}$ 29π



SOLN The three points define a triangle that is circumscribed by the circle. The side lengths of the triangle are $\sqrt{18}, \sqrt{98}$, and $\sqrt{116}$ showing that it is a right triangle. The circumcenter of a right triangle is the midpoint of the hypotenuse which is the point $(8, 1)$. The distance from the circle to any of the given points is $\sqrt{29}$ making the area 29π. □

28. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$ both be subsets of $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let X^C denote the complement in U of the subset X. What is $\left[[A \cap (A \cap B)^C] \cap B\right]^C$?

(A) $\{\}$ or $\varnothing$

$\boxed{\text{(B)}}$ $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

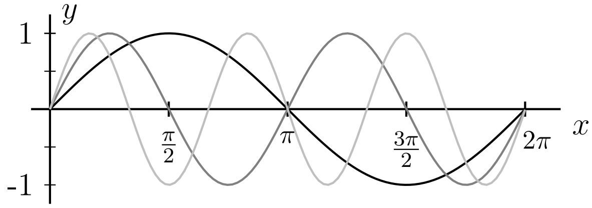
(C) $\{1, 3, 5, 7, 9\}$

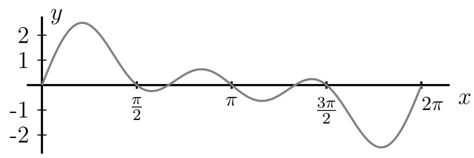
(D) $\{7, 8, 9\}$

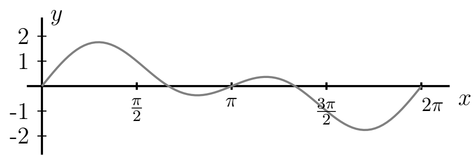
(E) $\{1, 3, 5\}$

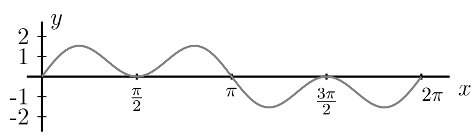
SOLN Working from the inside out, $A \cap B = \{2, 4, 6\}$ and $(A \cap B)^C = \{1, 3, 5, 7, 8, 9\}$. Then $A \cap (A \cap B)^C = \{1, 3, 5\}$. Intersecting this with B gives the empty set, whose complement is U. □

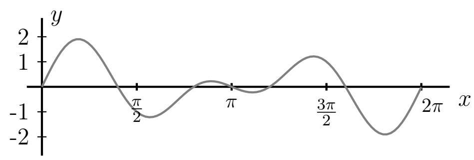
29. Below you will find graphs of $y = \sin x$, $y = \sin 2x$, and $y = \sin 3x$ on the interval $[0, 2\pi]$. Which of the following graphs show $y = \sin x + \sin 2x + \sin 3x$?

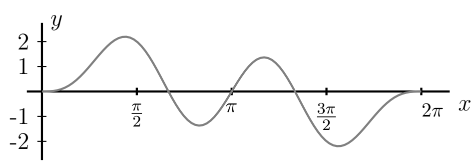


(A) 

(B) 

(C) 

(D) 

(E) 

SOLN For any x value, the y value of the sum is the sum of the y values of the individual functions. □

30. The *spiral of Theodorus* is built from an isosceles right triangle with legs of length 1. Additional right triangles are constructed with a leg of length one and the hypotenuse of the previous triangle serving as the other leg as illustrated below. Theodorus ended his construction with 16 triangles, although additional triangles may be added overlapping the prior triangles. If the first 100 triangles are constructed in this pattern, how many will have rational area?

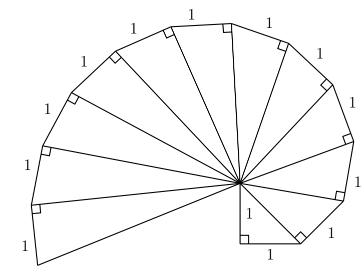
(A) 8

(B) 9

(C) 10

(D) 12

(E) 14



SOLN The first triangle has an area of $1/2$. Nine other triangles will have rational area if the leg length from the previous hypotenuse ($\sqrt{n-1}$) is rational. □

31. A cycloid is a curve traced out by a point on the rim of a wheel as it rolls without slipping along a flat surface. It is most easily defined by the set of parametric equations $x = r(t - \sin t), y = r(1 - \cos t)$. It can be shown that the area under one arch of a cycloid is $3\pi r^2$ where r is the radius of the wheel. The shaded part of the area under the cycloid shown below is $24\,\text{in}^2$. What is the area of the wheel?

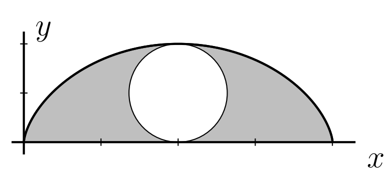
(A) $8\,\text{in}^2$

(B) $9\,\text{in}^2$

(C) $10\,\text{in}^2$

(D) $12\,\text{in}^2$

(E) $14\,\text{in}^2$



SOLN The shaded area is $3\pi r^2$ minus the area of the wheel (which is πr^2). Setting $2\pi r^2 = 24$, we get $r = \sqrt{12/\pi}$. Then $A_{\bigcirc} = \pi(\sqrt{12/\pi})^2 = 12\,\text{in}^2$. □

32. When each is given in Fahrenheit, the sum of the high and low temperatures for Ephraim on a particular day is 68. What is the sum of the high and low temperatures in Ephraim on the same day if each is given in Celsius? Hint: $F = \frac{9}{5}C + 32$.

$\boxed{\text{(A)}}$ $\frac{20}{9}$

(B) $\frac{36}{5}$

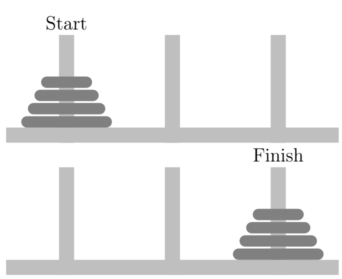
(C) 20

(D) $\frac{772}{9}$

(E) $\frac{772}{5}$

$\boxed{\textit{SOLN}}$ $F_{\text{hi}} + F_{\text{lo}} = 68$
$\left(\frac{9}{5}C_{\text{hi}} + 32\right) + \left(\frac{9}{5}C_{\text{lo}} + 32\right) = 68$
$\frac{9}{5}\left(C_{\text{hi}} + C_{\text{lo}}\right) = 68 - 32 - 32 = 4$ □

33. Four rings of different sizes are stacked on one of three posts in ascending order (smallest on top). You are able to move one ring at a time (taking the top ring from one post and moving it to another post), but you may never place a larger ring on a smaller ring. What is the minimum number of moves required to move the entire stack to a different post?



(A) 12

(B) 14

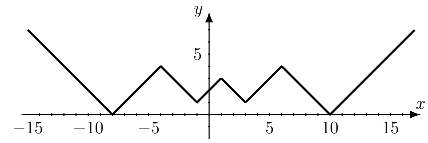
$\boxed{\text{(C)}}$ 15

(D) 16

(E) 17

$\boxed{\textit{SOLN}}$ This is the famous Towers of Hanoi game. Look for a pattern. To move one ring to a different post requires one move. Two rings require three moves, etc. n rings require $2^n - 1$ moves. □

34. Find the sum of all the real solutions to $|4 - |3 - |2 - |1 - x|||| = 0$.



(A) -2

(B) -1

(C) 0

(D) 1

$\boxed{\text{(E)}}$ 2

$\boxed{\textit{SOLN}}$ Outermost absolute value gives $4 = |3 - |2 - |1 - x|||$. This gives two equations: $4 = 3 - |2 - |1 - x||$ and $-4 = 3 - |2 - |1 - x||$. First has no solution.

Continuing with the second, we get $|2 - |1 - x|| = 7$ which breaks into $2 - |1 - x| = 7$ and $2 - |1 - x| = -7$. The first has no solution.

Continuing with the second, we get $|1 - x| = 9$ which gives $1 - x = 9$ and $1 - x = -9$ with solutions of $x = -8$ and 10. □

35. For real numbers a and b, define an operation Δ as $a \,\Delta\, b = ab^2 - |a|$. Find $[(-2)\,\Delta\,5]\,\Delta\,(-1)$.

$\boxed{\text{(A)}}$ -104

(B) -96

(C) -53

(D) 0

(E) 96

$\boxed{\textit{SOLN}}$ $((-2)(5)^2 - |-2|)(-1)^2 - \big|(-2)(5)^2 - |-2|\big| =$
$(-50 - 2)(1) - |-50 - 2| = -52 - 52 =$
-104 □

36. A bucket of 200 marbles is 1% green and 99% red. How many red marbles need to be removed so that the percentage of red marbles in the bucket will be 98%?

(A) 1

(B) 2

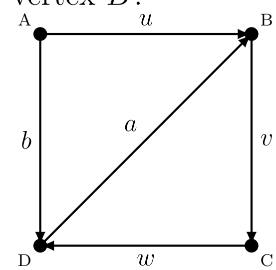
(C) 10

(D) 50

$\boxed{\text{(E)}}$ 100

$\boxed{\text{SOLN}}$ This is a reformulation of the counter-intuitive "potato paradox." Many people choose choice B, thinking that removing two red marbles will reduce the percentage of red by 1% as $2/200$ is 1%. However, that would leave $196/198$ red marbles, which is 98.99% red — far too large. Better is to think about how to get the green marbles from 1% to 2%. We can't add greens, so to double the greens as a percentage, we need to halve the total number of marbles. We can do this by removing 100 reds. □

37. A *directed graph* is a configuration of directed edges (arrows) and vertices (dots) such that vertices are connected to other vertices by a directed edge. A *directed path* follows the direction of any number (repeats possible) of directed edges that connect two vertices. The length of the path is equal to the number of edges in the path. For example, a directed path of length 4 from vertex A to vertex B has the following edge sequence (u, v, w, a). How many directed paths of length 48 are there from vertex A to vertex B?



$\boxed{\text{(A)}}$ 0

(B) 1

(C) 2

(D) 3

(E) 4

$\boxed{\text{SOLN}}$ There are 4 initial paths from vertex A to vertex B. They are (u), (u, v, w, a), (b, a), and (b, a, v, w, a). After following any of these initial paths, one ends up in a 3-cycle following edges (v, w, a). Thus, this question amounts to computing $(48 - 1) \mod 3$, $(48 - 4) \mod 3$, $(48 - 2) \mod 3$, or $(48 - 5) \mod 3$ and checking whether any of these equal 0. None of them do. Therefore, no directed path of length 48 between vertex A and vertex B exists, i.e., the answer is 0. □

38. Let $A_1, A_2, \ldots$ be a sequence of sets. Define $\bigcup_{k=1}^{\infty} A_k = \{x \mid x \in A_k \text{ for some } k\}$ to be an infinite union $A_1 \bigcup A_2 \bigcup A_3 \bigcup \cdots$, and $\bigcap_{k=1}^{\infty} A_k = \{x \mid \text{for every } k, x \in A_k\}$ to be an infinite intersection $A_1 \bigcap A_2 \bigcap A_3 \bigcap \cdots$. If $A_k = \{k, k+1, \ldots\}$, then what is $\bigcap_{k=1}^{\infty} A_k$?

(A) $\{1, 2, \ldots\}$

(B) $\{\infty\}$

(C) $\{k+1, k+2, \ldots\}$

(D) $\bigcup_{k=1}^{\infty} A_k$

$\boxed{\text{(E)}}$ $\varnothing$

$\boxed{\mathcal{SOLN}}$ Since k is not in A_{k+1}, k cannot be in the intersection. Therefore, there are no elements that are in all A_k. Choices A and D are equivalent. □

39. Tim is playing a game of dice. He must pay \$2 to roll a pair of fair six-sided dice. If he rolls doubles, he wins \$9. What are the expected net winnings per game?

$\boxed{\text{(A)}}$ −\$0.50

(B) −\$0.33

(C) \$0.50

(D) \$1.17

(E) \$3.00

$\boxed{\mathcal{SOLN}}$ Of the 36 possible outcomes, one-sixth of them (i.e., 6) are doubles.

$$\left(\frac{1}{6}\right)(9-2)+\left(\frac{5}{6}\right)(-2) = -\frac{3}{6} = -0.50$$

□

40. Find the limit.

$$\lim_{x\to\infty} \frac{16x^2 - x^3}{2^x}$$

(A) $-\infty$

(B) -1

$\boxed{\text{(C)}}$ 0

(D) 1

(E) ∞

$\boxed{\mathcal{SOLN}}$ Use L'Hopital's rule.

$$\lim_{x\to\infty} \frac{16x^2 - x^3}{2^x} = \lim_{x\to\infty} \frac{32x - 3x^2}{2^x \ln 2} =$$

$$\lim_{x\to\infty} \frac{32 - 6x}{2^x (\ln 2)^2} = \lim_{x\to\infty} \frac{-6}{2^x (\ln 2)^3}$$

□