Snow College Mathematics Contest

April 2, 2019 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. Find $\log_{10}(\log_{10}(\log_{10} 10^{10\,000\,000\,000}))$.

   (A) 0
   (B) 1
   (C) 2
   (D) 3
   (E) 6

2. In astronomy, the apparent brightness b of a star is related to the luminosity L by
$$b = \frac{L}{4\pi d^2}$$
where d is the distance to the star. If planet 1 is five times farther from a star than planet 2 is, what will be the ratio b_1/b_2?

   (A) $\frac{1}{25}$
   (B) $\frac{1}{5}$
   (C) 1
   (D) 5
   (E) 25

3. A semiprime is the product of two (not necessarily distinct) primes. They are very useful in cryptology because it is easy to multiply two primes together, but hard to factor a large semiprime. What is the sum of the semiprimes less than 20?

   (A) 45
   (B) 49
   (C) 54
   (D) 57
   (E) 58

4. Simplify:
$$\frac{\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}}{\log_2 \sqrt[4]{32\sqrt[3]{8}}}$$

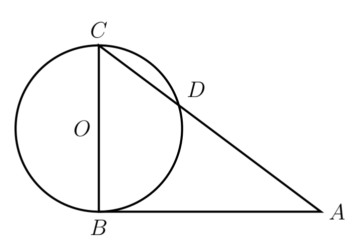
   (A) $\frac{5}{6}$
   (B) $\frac{3^{43/12}}{2^{3/2}}$
   (C) $\frac{\log_3 2}{\log_2 3}$
   (D) $\sqrt[3]{3}$
   (E) $\frac{43}{18}$

5. Let n be a positive integer; then n^2 is a perfect square. How many values of n exist such that $n^2 + 45$ is itself a perfect square?

   (A) 0
   (B) 1
   (C) 2
   (D) 3
   (E) 4

6. Circle with center O has radius 3, $AB = 8$, and $\overline{AB}$ is tangent to the circle at B. If $\overline{BC}$ is a diameter of the circle, find CD.

   (A) 3
   (B) 3.2
   (C) 3.6
   (D) 4
   (E) 6.4



7. What is the probability that the product of the numbers rolled on three fair six-sided dice is prime?
   (A) $\frac{1}{36}$
   (B) $\frac{1}{24}$
   (C) $\frac{1}{16}$
   (D) $\frac{1}{12}$
   (E) $\frac{1}{8}$

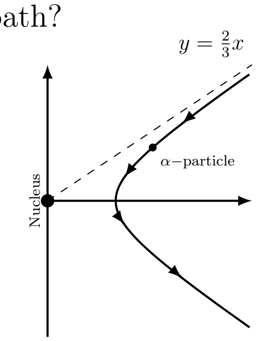
8. A triangle with integer side lengths and positive area has no two sides equal. Find its least possible perimeter.
   (A) 6
   (B) 7
   (C) 8
   (D) 9
   (E) 10

9. A positive integer x less than 35 satisfies the two congruences simultaneously:
$$x \equiv 2 \mod 5$$
$$x \equiv 3 \mod 7$$
   Which represents $2x \mod 35$?
   (A) -1
   (B) 0
   (C) 1
   (D) 2
   (E) 3

10. A die is *fair* if all faces are congruent and each number has an equal chance of being rolled. Which is not a possible number of faces for a fair die?
   (A) 4
   (B) 5
   (C) 8
   (D) 12
   (E) 20

11. The Dirac delta function is defined by
$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$
   and
$$\int_{-\infty}^{\infty} \delta(x)\ \mathrm{d}x = 1$$
   Which of the following functions would the Dirac delta function be a derivative of?
   (A) $p(x) = x!$
   (B) $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$
   (C) $g(x) = |x|$
   (D) $r(x) = \frac{1}{1-x}$
   (E) $s(x) = \frac{x-1}{1-x}$

12. In 1911, physicist Ernest Rutherford discovered that alpha particles shot toward the nucleus of an atom are repulsed away along hyperbolic paths. If a particle gets as close as 5 units to the nucleus with a slant asymptote of $y = \frac{2}{3}x$, what is the mathematical model representing the path?
   (A) $\frac{x^2}{25} - \frac{9y^2}{100} = 1$
   (B) $\frac{x^2}{5} - \frac{3y^2}{10} = 1$
   (C) $\frac{x^2}{9} - \frac{y^2}{4} = 1$
   (D) $\frac{x^2}{25} - \frac{y^2}{56.25} = 1$
   (E) $\frac{x^2}{3} - \frac{y^2}{2} = 1$



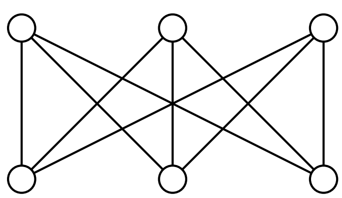
13. How many different ways can the letters in the word `STEMS` be arranged?
   (A) 16
   (B) 25
   (C) 45
   (D) 60
   (E) 120

14. Three boys entered a deli. The first ordered 4 sandwiches, a drink, and 10 donuts for \$8.45. The second ordered 3 sandwiches, a drink, and 7 donuts for \$6.30. How much did the third boy pay for a sandwich, a drink, and a donut?

    (A) \$2.00
    (B) \$2.05
    (C) \$2.10
    (D) \$2.15
    (E) \$2.20

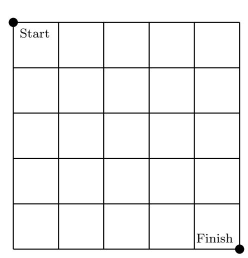
15. In *graph theory*, a graph is composed of vertices (dots) and edges (lines). What is the minimum number of vertex colors required so that no two connected vertices in the graph shown share the same color?

    (A) 1
    (B) 2
    (C) 3
    (D) 4
    (E) 5



16. Below is a 5×5 grid. A path from "Start" to "Finish" is a sequence of moves right or down along the grid lines in the figure. Left and up are not allowed. How many paths are there from "Start" to "Finish"?

    (A) $\binom{5}{2} \cdot \binom{5}{2}$
    (B) $\binom{10}{2} \cdot 5!$
    (C) $\binom{10}{5} \cdot \binom{10}{5}$
    (D) $5! \cdot 5!$
    (E) $\binom{10}{5}$



17. Let n be a nonnegative integer. The gamma function, $\Gamma(n)$, is a generalization of the factorial function $n!$. A recursive definition for the gamma function is $\Gamma(n+1) = n\Gamma(n)$, where $\Gamma(1) = 1$. Compute $\Gamma(5)$.

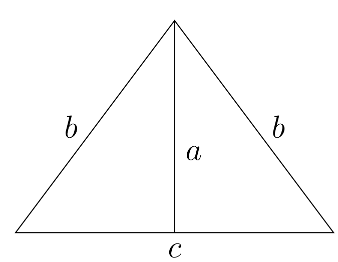
    (A) 6
    (B) 10
    (C) 15
    (D) 24
    (E) 120

18. Ted has a solid wooden cube with whole number dimensions (in centimeters). He paints the entire surface of the cube blue. Then, with slices parallel to the faces of the cube, Ted cuts it into $1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm}$ cubes. A certain number x of the small cubes are completely free of paint. A certain number y of the small cubes are painted on only one side. A certain number z of the small cubes are painted on two sides. Eight cubes have paint on three sides. If $y = 2x$, what was the side length of Ted's original cube?

    (A) 3 cm
    (B) 4 cm
    (C) 5 cm
    (D) 8 cm
    (E) 10 cm

19. The altitude a, equal sides b, and non-equal side c of an isosceles triangle have lengths that are, in the order listed, consecutive even numbers of centimeters. What is the area of the triangle?

    (A) $6\,\text{cm}^2$
    (B) $16\,\text{cm}^2$
    (C) $30\,\text{cm}^2$
    (D) $48\,\text{cm}^2$
    (E) $70\,\text{cm}^2$



20. The period T of a pendulum is the time it takes to make one complete oscillation. The period is related to the length L and the acceleration due to gravity $g = 9.8\,\text{m/s}^2$ by

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

How long should the pendulum be in order to have a period of $T = \pi$ seconds?

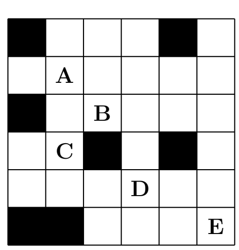
(A) $1\,\text{m}$
(B) $1.5\,\text{m}$
(C) $2.45\,\text{m}$
(D) $9.8\,\text{m}$
(E) $19.6\,\text{m}$

21. Find the value of C such that the parabola $y = x^2 + C$ is tangent to the line $y = x$.

(A) $\frac{1}{4}$
(B) $\frac{1}{2}$
(C) 1
(D) 2
(E) 4

22. In the grid of 1×1 squares, which of the squares A, B, C, D, or E, when blacked out will allow the white squares to be covered by exactly 14 dominoes (1×2 rectangles) with no overlaps or gaps?

(A) A
(B) B
(C) C
(D) D
(E) E



23. Eric filled 2/3 of his radiator with antifreeze and then added 4 more quarts (a gallon) of antifreeze. After draining half the antifreeze, he needed 11 quarts of antifreeze to fill the radiator to capacity. How many gallons of antifreeze can the radiator hold?

(A) 4.65
(B) 4.875
(C) 18.6
(D) 19.565
(E) 78

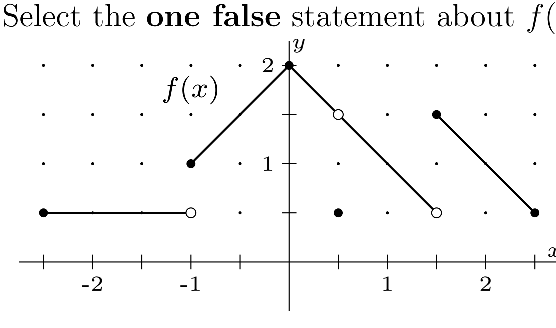
24. The complex Pauli spin matrices σ_1, σ_2, and σ_3 appear in quantum mechanics.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Which statement about all three σ_n is **not** true?

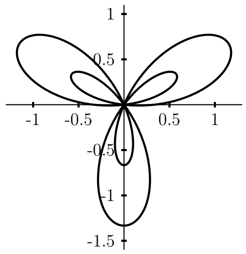
(A) $\sigma_n^3 = \sigma_n$
(B) $\det(\sigma_n) = -1$
(C) $\mathrm{Tr}(\sigma_n) = 0$
(D) σ_n is its own multiplicative inverse.
(E) The product of any two is the third.

25. Select the **one false** statement about $f(x)$.



(A) $f(-1) = 1$
(B) $\lim_{x \to -1} f(x) = 1$
(C) $\lim_{x \to 1/2} f(x) = \frac{3}{2}$
(D) $\lim_{x \to 3/2^-} f(x) = \frac{1}{2}$
(E) $\lim_{x \to 3/2^+} f(x) = \frac{3}{2}$

26. Which polar equation best represents the graph for $0 \le \theta \le 2\pi$?



(A) $r = \sin(3\theta) + \frac{1}{3}$

(B) $r = 6\theta$

(C) $r = \theta^3$

(D) $r = \cos(3\theta) - \frac{1}{3}$

(E) $r = \sin(6\theta)$

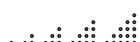
27. What is the output of the following BASIC computer program with some natural number n given as input?

```
10 input n
20 t = 0
30 for i = 1 to n
40 t = t + i
50 next i
60 print t
```

(A) $n!$

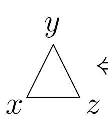
(B) the prime factors of n

(C) least common multiple of n and f

(D) nth triangular number 

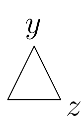
(E) n^n

28. Definition of the *triangle of power* notation:



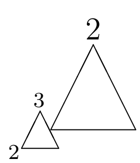
$\Leftrightarrow x^y = z \Leftrightarrow \sqrt[y]{z} = x \Leftrightarrow \log_x z = y$

Any of x, y, z is equivalent to the triangle of power with that number missing, e.g.,



$= x$

Find the value of the following.



(A) $\sqrt{2}$

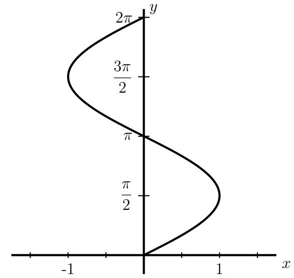
(B) $\sqrt[3]{8}$

(C) 3

(D) $\sqrt{8}$

(E) 64

29. Which set of parametric equations will produce the graph shown?



(A) $\begin{cases} x(t) = \sin t \\ y(t) = \cos t \end{cases}$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

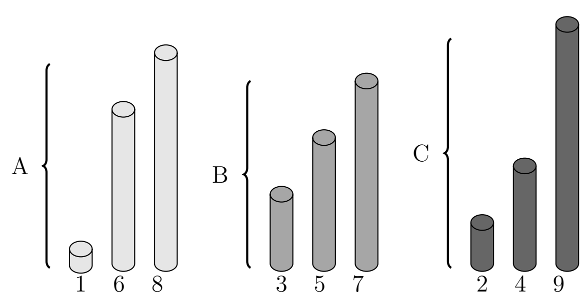
(B) $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

(C) $\begin{cases} x(t) = t \\ y(t) = \cos t \end{cases}$ $\quad -1 \le t \le 1$

(D) $\begin{cases} x(t) = \sin 2t \\ y(t) = \cos t \end{cases}$ $\quad 0 \le t \le 2\pi$

(E) $\begin{cases} x(t) = \sin t \\ y(t) = t \end{cases}$ $\quad 0 \le t \le 2\pi$

30. A binary relation $\mathcal{R}$ on a set X is *transitive* if for all $a, b, c \in X$, if $a \mathcal{R} b$ and $b \mathcal{R} c$, then $a \mathcal{R} c$. An example is the $<$ relation on $\mathbb{N}$. Rock, Paper, Scissors is **not** transitive. For individual sticks, "longer than" is transitive; but for **sets** of sticks, the relation "longer than more often" (denoted $\succ$) doesn't need to be. Give the relationships for these sets.



(A) $C \succ A \succ B \succ C$

(B) $B \succ A \succ C \succ B$

(C) $C \succ B \succ A \succ C$

(D) $B \succ A \succ C \succ B$

(E) $A \succ C \succ B \succ A$

31. All numbers in this problem are in base 7. What is $461 + 246$?

(A) 640

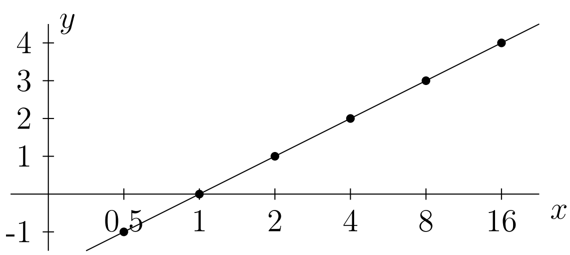
(B) 645

(C) 707

(D) 1040

(E) 1160

32. Which equation best represents the graph?



(A) $y = \sqrt{x}$

(B) $y = 2^x$

(C) $y = \log_2 x$

(D) $y = 2x + 1$

(E) $y = 2x - 2$

33. The sum of three consecutive integers is equal to the product of those integers. How many sets of integers satisfy this condition?

(A) 0

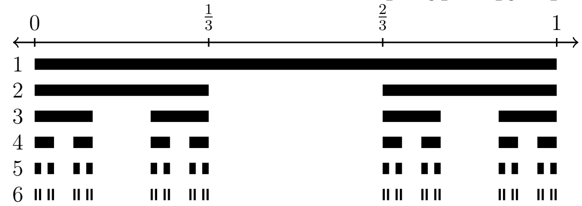
(B) 1

(C) 2

(D) 3

(E) 5

34. The Cantor set is constructed by removing in an infinite number of steps the open middle third of each remaining interval of $[0, 1]$. For example, the second step is $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.



Which is **not** a property of the Cantor set?

(A) It has no intervals.

(B) Between any two numbers in the set, there is a number not in the set.

(C) It has a finite number of points.

(D) It is self-similar at every scale.

(E) It consists of numbers that, in base 3, contain only 0s and 2s.

35. Find the value of the following sum:

$$\sum_{k=1}^{2019} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

(A) $\frac{2016}{2019}$

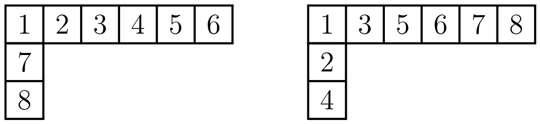
(B) $\frac{2018}{2019}$

(C) $\frac{2019}{2020}$

(D) 1

(E) $\frac{2020}{2019}$

36. The following two diagrams are examples of a type of *Young tableaux*.



Diagrams of this type are often used to count complex algebraic symmetries. The rules:

- All integers 1 through 8 must appear in the 8 boxes.
- The numbers in the row must increase to the right
- the numbers in the column must increase downward.

How many possible Young tableaux of this shape are there that follow these rules?

(A) 18

(B) 21

(C) 24

(D) 32

(E) 35

37. Lines are drawn from point E to points B, C, and D in adjacent squares, creating angles α, β, and γ as shown. What is the sum of α, β, and γ?

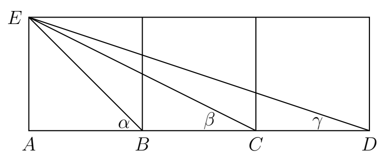
(A) 50°

(B) 60°

(C) 70°

(D) 80°

(E) 90°



38. In the sequence, how many black balls are needed for the 100th term?

(A) 200

(B) 202

(C) 201

(D) 198

(E) 199

39. Which of the following is NOT equivalent to the others?

(A) $\tan^2 x$

(B) $\frac{\sin^2 x}{\cos^2 x}$

(C) $\frac{1-\cos 2x}{1+\cos 2x}$

(D) $\sec^2 x - 1$

(E) $2 - 2\cos^2 x$

40. On a standardized test, Jan scores 70 points. Scores are normally distributed with a mean of 50 and a standard deviation of 20. Which is closest to Jan's percentile?

(A) 20th

(B) 50th

(C) 70th

(D) 85th

(E) 100th