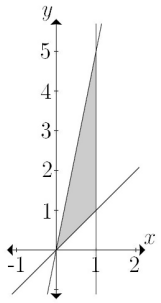
Snow College Mathematics Contest

April 4, 2017 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. Find the volume of the 3-D shape formed by revolving the region bounded by $y = 5x$, $y = x$, and $x = 1$ around the y-axis.



   (A) $\frac{8\pi}{3}$
   (B) 24π
   (C) $\frac{4\pi}{3}$
   (D) 16π
   (E) $\frac{16\pi}{3}$

2. If $a + b = 3$ and $a^2 + b^2 = 89$, then what is $a^3 + b^3$?

   (A) 307
   (B) 347
   (C) 387
   (D) 507
   (E) Not possible to determine

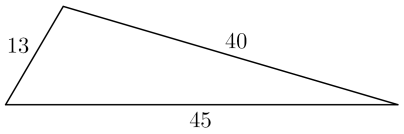
3. What is the value of $\log_2 \left(7^{-\log_7 \frac{1}{8}}\right)$?

   (A) -3
   (B) $\frac{1}{8}$
   (C) $\frac{1}{3}$
   (D) 3
   (E) 8

4. How many parallelograms can be formed with these three points as vertices, and what is the area of each? $(0, 0)$, $(1, \sqrt{3})$, $(4, 0)$

   (A) 1, $A = 5\sqrt{3}$
   (B) 3, $A = 4\sqrt{3}$
   (C) 4, $A = 10$
   (D) 2, $A = 5\sqrt{3}$
   (E) 2, $A = 8$

5. Find the area of the triangle.



   (A) 252
   (B) 260
   (C) 298
   (D) 312
   (E) 333

6. The operation $\oplus$, called *reciprocal sum*, is useful in many areas of physics. $x = a \oplus b$ means $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$. Compute the following.
   $4 \oplus 2 \oplus 4 \oplus 3 \oplus 4 \oplus 4 \oplus 2 \oplus 3 \oplus 2 \oplus 4 \oplus 4 \oplus 3$

   (A) $\frac{3}{4}$
   (B) $\frac{1}{2}$
   (C) $\frac{3}{8}$
   (D) $\frac{1}{8}$
   (E) $\frac{1}{4}$

7. Both the surface area and volume of a sphere increase as the radius increases. A increases faster than V when r is small, but V increases faster than A when r is large. At what value of r does the rate of increase in V overtake the rate of increase in A?

   (A) $\frac{2}{3}$
   (B) 1
   (C) $\frac{4}{3}$
   (D) 2
   (E) $\frac{8}{3}$

8. In base three, what is three times 2102?

   (A) 20210
   (B) 2110
   (C) 12100
   (D) 11201
   (E) 21020

9. An earthquake generates both *primary* and *secondary* seismic waves. In a certain material the longitudinal P-waves have a speed of 5000 m/s and the transverse S-waves have a speed of 3000 m/s. If a seismometer detects P-waves in that material 4 s before the S-waves, how far away is the epicenter?

   (A) 20 km
   (B) 30 km
   (C) 40 km
   (D) 50 km
   (E) 60 km

10. For a matrix A, an eigenvalue-eigenvector pair is a constant λ and corresponding nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Which is an eigenvalue λ of this matrix?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

    (A) -1
    (B) 0
    (C) 1
    (D) 2
    (E) 4

11. Definition of the *triangle of power* notation:

$$\begin{array}{ccc} & y & \\ x & \triangle & z \end{array} \Leftrightarrow x^y = z \Leftrightarrow \sqrt[y]{z} = x \Leftrightarrow \log_x z = y$$

    Any of x, y, z is equivalent to the triangle of power with that number missing, e.g.,

$$\begin{array}{ccc} & y & \\ & \triangle & \\ & z & \end{array} = x$$

    Find the value of the following.

$$\begin{array}{ccc} & 2 & \\ 2 & \triangle & \\ 2 & & \end{array}$$

    (A) 2
    (B) $\sqrt{8}$
    (C) 3
    (D) 4
    (E) 16

12. The city council has changed the numbering scheme for the 200 houses on Elm Street. They will be renumbered with the natural numbers from 1 through 200. A city worker is given a box of 1000 metal numbers, 100 of each digit, and told to distribute new house numbers in order starting with 1 Elm Street. What is the first address for which he will not have the correct digits?

    (A) 137 Elm Street
    (B) 163 Elm Street
    (C) 172 Elm Street
    (D) 191 Elm Street
    (E) 199 Elm Street

13. What is the smallest positive integer n such that $1 + 2 + 3 + \ldots + n > 5000$?

    (A) 90
    (B) 99
    (C) 100
    (D) 101
    (E) 110

14. A *metric space* is a set A where we have a notion of distance, i.e., if $a, b \in A$, then $d(a, b)$ is the "distance" between a and b. The distance function must satisfy these conditions:

(i) $d(a, b) \geq 0$ for all $a, b \in A$

(ii) $d(a, b) = 0$ iff $a = b$

(iii) $d(a, b) = d(a, b)$

(iv) $d(a, c) \leq d(a, b) + d(b, c)$

Does $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ define a metric on $\mathbb{R}^2$ (the xy-plane)?

(A) No; condition (i) is not met.

(B) No; condition (ii) is not met.

(C) No; condition (iii) is not met.

(D) No; condition (iv) is not met.

(E) Yes; all conditions are met.

15. What is the ratio of the area of a circle to the area of the figure created by flipping each quarter circle around its chord?

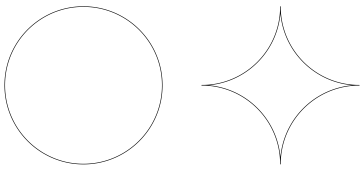
(A) $\frac{3\pi}{4}$

(B) $\pi + 1$

(C) π

(D) $\pi - 1$

(E) $\frac{\pi}{4-\pi}$



16. How many times does 12 appear in the output of this BASIC computer program?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

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10 for i = 1 to 4
20   for j = 1 to i
30    for k = 1 to j
40     print i*j*k
50    next k
60   next j
70 next i
```

17. In geometry, a *kissing number* is the number of non-overlapping unit spheres that can be arranged such that they each touch a given unit sphere. In 1-D, the kissing number is 2. In 3-D the kissing number is 12. What is the kissing number in 2-D?

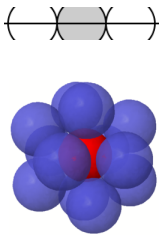
(A) 4

(B) 5

(C) 6

(D) 7

(E) 8



18. Given a triangle of base b and altitude h, a rectangle of height x is inscribed in the triangle with its base in the base of the triangle. What is the area of the rectangle?

(A) $\frac{bx}{h}(h - x)$

(B) $\frac{hx}{b}(b - x)$

(C) $\frac{bx}{h}(h - 2x)$

(D) $x(b - x)$

(E) $x(h - x)$

19. Simplify $\sqrt{\sqrt[0.06]{2^{0.12}}}$

(A) 1

(B) $\sqrt{2}$

(C) 2

(D) $2^{0.0384}$

(E) 4

20. Find the remainder for the quotient.

$$\frac{2x^3 - 9x^2 + 6x - 1}{2x - 1}$$

(A) $2x - 1$

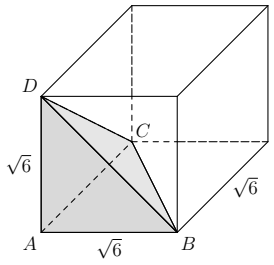
(B) $2x^2 - 8x + 2$

(C) 0

(D) $\frac{1}{2}$

(E) $2x^3 - 9x^2 + 3$

21. A cube with side length $\sqrt{6}$ cm is cut revealing pyramid $ABCD$ as shown. Find the surface area of the pyramid.



(A) 12
(B) 18
(C) $4\sqrt{6}$
(D) $9 + 3\sqrt{3}$
(E) $54 + 3\sqrt{3}$

22. In a drawer are 6 black, 2 gray, and 2 tan socks. What is the probability of blindly pulling out (without replacement) four black socks in a row?

(A) $\frac{1}{14}$
(B) $\frac{1}{7}$
(C) $\frac{1}{4}$
(D) $\frac{2}{5}$
(E) $\frac{3}{5}$

23. Here is an **incorrect** "proof" that $1 = -1$. Which step of the proof is incorrect?

$1 = \sqrt{1}$
(A) $1 = \sqrt{(-1)(-1)}$
(B) $1 = \sqrt{-1}\sqrt{-1}$
(C) $1 = i \cdot i$
(D) $1 = i^2$
(E) $1 = -1$

24. Given sets $A = \{\text{evens}\}$, $B = \{\text{non-primes}\}$, and $C = \{\text{primes} < 19\}$, and universal set $U = \{0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23\}$, find the complement of $A \cup B \cup C$.

(A) $\{0, 23\}$
(B) $\{0, 10, 20\}$
(C) $\{23\}$
(D) $\{1, 3, 11, 13, 21, 23\}$
(E) $\emptyset$ or $\{\ \}$

25. In a music class, 15 students play violin, 14 play piano, and 16 play horns. Of these, 6 play piano and violin, 9 play piano and horns, and 5 play horns and violin. Four students play all three and 2 students play none. How many students are in the class?

(A) 21
(B) 23
(C) 31
(D) 45
(E) 47

26. A cyclist made a 31.5 mile trip. Later, she calculated that if she had increased her average speed by 2 mph, she would have saved an hour. What was her original speed?

(A) 6 mph
(B) 7 mph
(C) 8 mph
(D) 9 mph
(E) 10 mph

27. What is the sum of the real solutions?

$$\left(\frac{x-5}{3}\right)^{x^2+x} = 1$$

(A) 9
(B) -1
(C) 8
(D) 1
(E) 10

28. The average of a and $2b$ is 7; the average of a and $2c$ is 8. What is the average of a, b, and c?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 9

29. For how many integers b is the polynomial $x^2+bx+16$ factorable over the integers?

(A) 2
(B) 3
(C) 4
(D) 5
(E) 6

30. A student committee must consist of two seniors and three juniors. Five seniors are available to serve. What is the least number of junior volunteers needed if the selectors want at least 600 different possible ways to choose the committee?

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

31. Square $PQRS$ has sides of length 10. Points T, U, V, and W are chosen on sides PQ, QR, RS, and SP respectively so that $PT = QU = RV = SW = 2$. Find the area of quadrilateral $TUVW$.

(A) 48
(B) 52
(C) 56
(D) 64
(E) 68

32. Let
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{10}$$
Find the last two digits of element b.

(A) 15
(B) 23
(C) 31
(D) 63
(E) 07

33. Find the sum of the series:
$$\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$$

(A) $\frac{3}{2}$
(B) ∞
(C) π
(D) $\sqrt{2}$
(E) $\frac{9}{4}$

34. Two straight lines have the same y-intercept and reciprocal slopes. If the first line has slope m and x-intercept a, what is the x-intercept of the other line?

(A) $\frac{m^2}{a}$
(B) $\frac{m}{a}$
(C) $\frac{1}{a}$
(D) am
(E) am^2

35. In one game you pick three distinct numbers from 01 to 20. You win if the three numbers you pick match the three numbered balls drawn at random (order doesn't matter). What is the probability of winning?

(A) $\frac{3}{20}$
(B) $\frac{3!\,17!}{20!}$
(C) $\frac{1}{17!}$
(D) $\frac{3!}{17!}$
(E) $\frac{20!}{3!\,17!}$

36. Pi High School sent a team of 5 students to the Snow College Math Contest in 2015. In 2016 PHS sent the same team except the oldest member (graduated) was replaced with a younger student. If the average team member age was the same for both years, how many years younger was the new member than the old member who was replaced?

    (A) 1

    (B) 2

    (C) 3

    (D) 4

    (E) 5

37. A regular decagon, side length 1, is inscribed in a circle. Find the radius of the circle.

    (A) 1

    (B) $\frac{1+\sqrt{5}}{2}$

    (C) $\frac{1-\sqrt{5}}{2}$

    (D) $\frac{\sqrt{5}}{2}$

    (E) $\frac{\sqrt{10}}{2}$

38. What is the range of possible values for the sum S of angles of a triangle drawn on the surface of a sphere?

    (A) $S = 180°$

    (B) $180° < S < 360°$

    (C) $180° < S < 720°$

    (D) $180° < S < 900°$

    (E) $270° < S < 1080°$

39. Adam celebrates only his prime birthdays. Next year there will be a celebration. Adam most recently celebrated his birthdays 5 and 7 years ago. If Adam is currently 78, how many times prior to this year has the preceding information been true?

    (A) never

    (B) once

    (C) twice

    (D) thrice

    (E) more than three times

40. John has a weird way of cutting pie. The first piece is a full half of the pie. The second piece is 1/3 of the remaining half of pie. Then the 3rd piece is 1/4 of the remainder and so on. What fraction of the original pie is the 5th piece?

    (A) $\frac{1}{720}$

    (B) $\frac{1}{30}$

    (C) $\frac{1}{120}$

    (D) $\frac{1}{6}$

    (E) $\frac{1}{540}$