Snow College Mathematics Contest

Bubble in the single best choice for each question you choose to answer.

1. Which of the products is/are palindromic (read the same backwards and forwards)?
   - (i) 11111×11111
   - (ii) 22222×22222
   - (iii) 33333×33333
   - (iv) 44444×44444

   (A) only (i)

   (B) only (ii) and (iii)

   (C) only (iii) and (iv)

   (D) only (i), (ii), and (iv)

   (E) all of them

2. If $\sqrt{x} + \sqrt{x+7} = 7$, then what is the value of $2\sqrt{x-5} + \sqrt{x-8}$?

   (A) 5

   (B) 8

   (C) 11

   (D) 14

   (E) $2\sqrt{3}$

3. The Pauli spin matrices σ_1, σ_2, and σ_3 appear in quantum mechanics. They are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
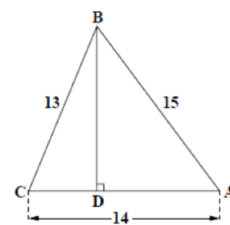
   What is their common determinant?

   (A) 0

   (B) -1

   (C) i

   (D) $-i$

   (E) I

4. $\log_a 2 = 0.356$ and $\log_a 3 = 0.565$ for some mystery base a. Compute $\log_a 12$.

   (A) 0.072

   (B) 0.202

   (C) 0.922

   (D) 1.277

   (E) 1.487

5. For $\triangle ABC$, $BC = 13$, $AC = 14$, $AB = 15$. D is the point on $\overline{AC}$ such that $\overline{BD}$ is perpendicular to $\overline{AC}$. Find the length of $\overline{BD}$.

   (A) 11

   (B) $11\frac{2}{3}$

   (C) 12

   (D) $12\frac{1}{4}$

   (E) $12\frac{1}{3}$

6. In all countries but the USA and Canada paper sizes are such that a larger sheet made from two equal sheets of the next smaller size has the same aspect ratio as the smaller sheets. What is that aspect ratio?

   (A) $\sqrt{2} : 1$

   (B) $2 : 1$

   (C) $3 : 2$

   (D) $\frac{1+\sqrt{5}}{2} : 1$

   (E) $8 : 5$
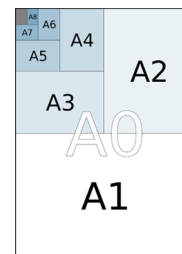
7. If $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ is the factorial, and the double factorial is

$$n!! = \begin{cases} n \cdot (n-2) \ldots 5 \cdot 3 \cdot 1 & n > 0 \text{ odd} \\ n \cdot (n-2) \ldots 6 \cdot 4 \cdot 2 & n > 0 \text{ even} \\ 1 & n = -1, 0 \end{cases}$$
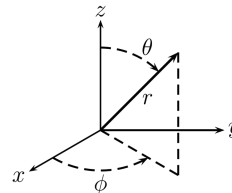
Which of the following statements are true?

(i) $(2n)!! = 2^n\, n!$

(ii) $(2n+1)!!\, 2^n\, n! = (2n+1)!$

(iii) $n! = n!!\,(n-1)!!$

(iv) $n!! = (n!)!, \quad n > 2$

(A) only (i) and (ii)

(B) only (ii) and (iii)

(C) only (iii) and (iv)

(D) only (i), (ii), and (iii)

(E) all of them

8. When 15 is added to a set of ten numbers, the median changes from 6 to 8. Find the median of the new set if 7 replaces 15.

(A) 4

(B) 5

(C) $5\frac{1}{2}$

(D) 6

(E) 7

9. A box contains two coins: one has heads on both sides; the other is a regular coin. A coin is selected at random and one side is observed to be heads. What is the probability that the other side is also heads?

(A) 2/3

(B) 1/4

(C) 3/4

(D) 1/3

(E) 5/8

10. Spherical coordinates (r, θ, ϕ) are defined as shown. Which ranges of variables would cover exactly once all points inside a solid sphere of radius a centered at the origin?

(A) $0 \le r < a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$

(B) $0 \le r < \frac{a}{2}, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$

(C) $0 \le r < a, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}$

(D) $0 \le r < a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \pi$

(E) $0 \le r < a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi$

11. Say you place a 25 000-mile-long metal band snugly around the earth's equator. (Assume a smooth spherical earth.) Then you cut the band and splice another 50 feet to it, thus loosening it all around. What is the tallest object that could comfortably fit between the new-length band and the earth?

(A) a DNA molecule

(B) a grain of sand

(C) a golf ball

(D) a small dog

(E) a tall person

12. Define $a\#b = ab^2 + a$ for integers $a, b > 0$. If $(a\#b)\#3 = 250$, find $a + b$.

(A) 6

(B) 7

(C) 8

(D) 9

(E) 10

13. Express the continued fraction expression for x as a simple closed-form number.

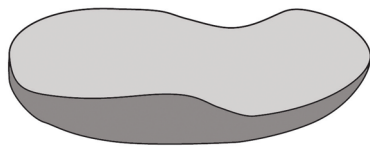$$x = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}$$

(A) $\sqrt{5}$

(B) $\frac{1-\sqrt{2}}{2}$

(C) $\frac{1+\sqrt{2}}{2}$

(D) $2 + \frac{\sqrt{2}}{2}$

(E) $\frac{4}{\sqrt{2}}$

14. Pick any odd number greater than one. Subtract 1 from the square of that odd number. What is the greatest positive integer that must be a divisor of the result?

(A) 2

(B) 3

(C) 4

(D) 8

(E) 16

15. A 1/50 scale model of a pond is shown. If the volume of the model is $40\,\text{cm}^3$, then what is the volume of the actual pond?

(A) $5000\,\text{m}^3$

(B) $1.25\,\text{m}^3$

(C) $5\,\text{m}^3$

(D) $500\,\text{m}^3$

(E) $20\,\text{m}^3$

16. The graph of the equation $|x|^n + |y|^n = 1$ with $n = 0.5$ is shown. As n changes, so does the shape of the graph; it is a diamond with vertices on the axes when $n = 1$. What shape does the graph approach as $n \to \infty$?

(A) unit circle

(B) four-leaf clover

(C) square

(D) five-pointed star

(E) two ellipses
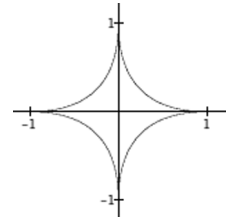
17. A machine depreciates by $\frac{1}{5}$ of its current value each year. If it costs $450 new, what is its value after 2 years?

(A) $360

(B) $288

(C) $230

(D) $184

(E) $147

18. A projectile is fired straight up so that its height in feet above the ground t seconds after firing is $s(t) = -16t^2 + 80t + 96$. Find the maximum height reached and how long it takes to reach that height.

(A) 180 feet; 6 seconds

(B) 180 feet; 3.5 seconds

(C) 196 feet; 2.5 seconds

(D) 132 feet; 0.5 seconds

(E) 196 feet; 1 second

19. A rectangular piece of sheet metal has a length that is 6 in less than twice the width. A 3 in × 3 in square piece is cut from each corner. The sides are then turned up to form an uncovered box of volume 150 in³. Find the dimensions of the original piece.

(A) $w = 4.5$ in, $l = 9$ in

(B) $w = 8$ in, $l = 10$ in

(C) $w = 10.5$ in, $l = 15$ in

(D) $w = 7$ in, $l = 8$ in

(E) $w = 11$ in, $l = 16$ in

20. Jeff paddles his canoe upstream for 3 miles and then returns to his original location. The round-trip takes 2 hours. If the current of the river is 2 mph, how fast does Jeff row his canoe in still water?

(A) 1 mph

(B) 2 mph

(C) 3 mph

(D) 4 mph

(E) 5 mph

21. Find the value of c so that the equation will have exactly one rational solution.
$$144x^2 - 216x + c = 0$$

(A) 9

(B) 12

(C) 24

(D) 81

(E) 108

22. Simplify. $\left(\dfrac{4-i}{1+i} - \dfrac{2i}{2+i}\right) 4i$

(A) $\frac{1}{3} - \frac{3}{2}i$

(B) $\frac{7}{10} - \frac{21}{10}i$

(C) $\frac{9}{10} - \frac{27}{10}i$

(D) $\frac{66}{5} + \frac{22}{5}i$

(E) $11 - 11i$

23. An n-dimensional hypersphere of radius r has a volume of
$$V_n(r) = r^n \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

where the gamma function is given by
$$\Gamma(n+1) = n\Gamma(n)$$
with $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Find the volume of a 4-D hypersphere of radius 2.

(A) $8\pi^2$

(B) $8\pi^4$

(C) $4\pi^2$

(D) $\pi^4/2$

(E) $4\pi^4$

24. What is the value of the following sum?
$$2 + 4 + 6 + 8 + \ldots + 2008 + 2010 + 2012$$

(A) 506520

(B) 1013042

(C) 2026084

(D) 3039126

(E) 4052168

25. How many distinct solutions does the equation have in the half-open interval $(0, \pi]$?
$$\sin(x) \cdot \sin(2x) \cdot \sin(3x) \cdot \ldots \cdot \sin(6x) = 0$$

(A) 10

(B) 12

(C) 15

(D) 18

(E) 21

26. Which is an *anti*derivative of $\sin^2\theta$?
    (A) $2\sin\theta\cos\theta$
    (B) $\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$
    (C) $\frac{1}{3}\sin^3\theta$
    (D) $\frac{1}{2}\theta + \frac{1}{4}\cos(2\theta)$
    (E) $-\frac{1}{4}\sin(2\theta)$

27. What is the coefficient of the x^7y^3 term in the expansion of $(x+y)^{10}$?
    (A) 120
    (B) 210
    (C) 720
    (D) 360
    (E) 240

28. What is the *range* of $f(x) = \frac{3}{1-e^{2x}}$?
    (A) $(0,3)$
    (B) $(-\infty,0) \cup (3,\infty)$
    (C) $(3,\infty)$
    (D) $(-\infty,-3) \cup (0,\infty)$
    (E) $f(x) \neq 0, 3$

29. The sum of two numbers is 10; their product is 20. Find the sum of their reciprocals.
    (A) $\frac{1}{10}$
    (B) $\frac{1}{2}$
    (C) 1
    (D) 2
    (E) 4

30. If the margin M (defined as selling price minus cost) made on an article costing C dollars and selling for S dollars is $M = \frac{1}{n}C$, then find the margin in terms of S.
    (A) $M = \frac{1}{n-1}S$
    (B) $M = \frac{1}{n}S$
    (C) $M = \frac{n}{n+1}S$
    (D) $M = \frac{1}{n+1}S$
    (E) $M = \frac{n}{n-1}S$

31. Find the solution set.
    $$\frac{7}{m+4} - \frac{6}{m-4} = \frac{-56}{m^2-16}$$
    (A) $\{-4\}$
    (B) $\{7\}$
    (C) $\{4\}$
    (D) $\emptyset$
    (E) $\{\text{all real numbers}\}$

32. A student on vacation for d days observed that (1) it rained 7 times, morning or afternoon, (2) when it rained in the afternoon it was clear in the morning, (3) there were 5 clear afternoons, and (4) there were 6 clear mornings. What is d?
    (A) 7
    (B) 9
    (C) 10
    (D) 11
    (E) 12

33. A right circular cone has for its base a circle having the same radius as a given sphere. The volume of the cone is one-half that of the sphere. What is the ratio of the altitude of the cone to the radius of the base?
    (A) 1/1
    (B) 1/2
    (C) 2/3
    (D) 2/1
    (E) $\sqrt{5/4}$

34. Assume the following 3 statements are true:
    - All teenagers are human.
    - All students are human.
    - Some students think.

    Which of the following are logical consequences of the above statements?

    (i) All teenagers are students.

    (ii) Some humans think.

    (iii) No teenagers think.

    (iv) Some humans who think are not students.

    (A) (ii)

    (B) (iv)

    (C) (ii), (iii)

    (D) (ii), (iv)

    (E) (i), (ii)

35. Find the 100th digit after the decimal point in $0.\overline{341729}$.
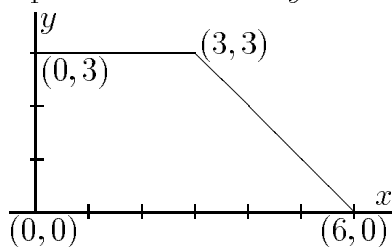
    (A) 1

    (B) 2

    (C) 3

    (D) 4

    (E) 7

36. My pet rabbit, Cotton, can hop up one step at a time or two steps at a time. The stairs in my house have ten steps. How many ways can Cotton get up my stairs?

    (A) 20

    (B) 32

    (C) 89

    (D) 117

    (E) 1024

37. Find the volume of the solid obtained by rotating the trapezoid around the y-axis.

    (A) $\frac{243}{4}\pi$

    (B) 63π

    (C) 72π

    (D) 90π

    (E) 108π

38. A basketball fieldhouse seats 15 000. Court-side seats sell for $9, baseline for $7, and balcony for $4. The total revenue for a sell-out is $81 000. If half the courtside and balcony seats and all the baseline seats are sold the total revenue is $47 500. How many of each type of seat are there?

	courtside	baseline	balcony
(A)	4000	3000	8000
(B)	3200	1800	10 000
(C)	3000	3000	8000
(D)	3000	2000	10 000
(E)	3500	2500	9000

39. Which equation best matches the graph?
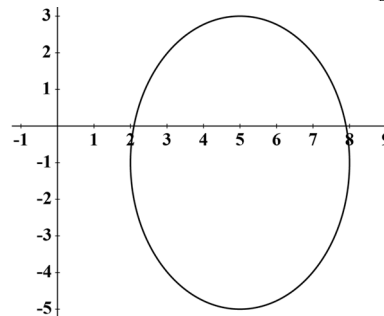
    (A) $\frac{(x-5)^2}{9} - \frac{(y+1)^2}{16} = 1$

    (B) $\frac{(x-5)^2}{16} + \frac{(y+1)^2}{9} = 1$

    (C) $\frac{(y+1)^2}{16} - \frac{(x+5)^2}{9} = 1$

    (D) $\frac{(x+5)^2}{9} + \frac{(y-1)^2}{16} = 1$

    (E) $\frac{(x-5)^2}{9} + \frac{(y+1)^2}{16} = 1$

40. How many 3-digit numbers that contain three different digits are between 300 and 800 and use only 1, 2, 3, 4, 5, 6, 7, 8, 9?

    (A) 280

    (B) 336

    (C) 360

    (D) 405

    (E) 440