Snow College Jr. Mathematics Contest

key

April 7, 2015 **Junior Division: Grades 7–9** **Form: T**

Bubble in the single best choice for each question you choose to answer.

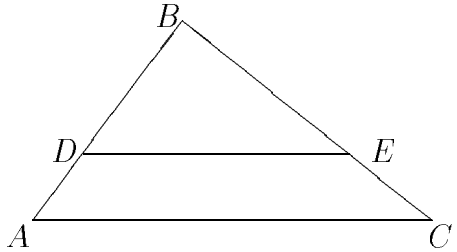
1. Chuck, Jackie, and Bruce are all celebrating their birthdays on the same day. Bruce's present age is seven years less than the sum of Chuck's and Jackie's present ages. In five years, Bruce will be twice as old as Jackie will be then. Three years ago, Jackie was one-third as old as Chuck was. What is the sum of all of their current ages?

   (A) 25
   (B) 31
   (C) 39
   (D) 48
   (E) 53

   SOLN From the statements given, we create the following system of equations: (1) $b = (c + j) - 7$, (2) $b + 5 = 2(j+5)$, and (3) $j - 3 = \frac{1}{3}(c-3)$. From (2) we get $b = 2j + 5$, and from (3) we get $c = 3j - 6$. By substitution (1) becomes $(2j+5) = (3j-6) + j - 7$. Solving yields $j = 9$, $c = 21$, and $b = 23$. Thus, the sum of current ages is 53. □

2. In the figure, $\overline{DE}$ is parallel to $\overline{AC}$, $m\angle ABC = x + 13°$, $m\angle BDE = 2x + 3°$, and $m\angle ACB = x°$. Find $m\angle ACB$.

   (A) 39°
   (B) 41°
   (C) 45°
   (D) 49°
   (E) 51°



   SOLN Because $\overline{DE} \parallel \overline{AC}$, $m(\angle BDE) = m(\angle BAC)$. Since $m(\angle ABC) + m(\angle ACB) + m(\angle BAC) = 180°$, then $(x + 13)° + x° + (2x + 3)° = 180°$. □

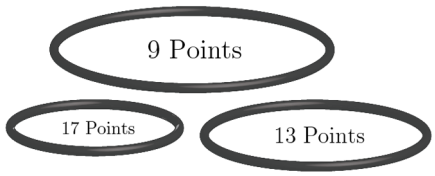
3. If you place one penny on the first square of an 8×8 checkerboard, two pennies on the second square, four pennies on the third square, and keep doubling the number of pennies for each square, how many total pennies will be on the entire board?

   (A) 2^{64}
   (B) 2^{63}
   (C) $2^{63} + 2^{62}$
   (D) 2^{65}
   (E) $2^{64} - 1$

   SOLN The number of pennies forms a geometric series $1 + 2 + 4 + \ldots + 2^{63}$. Summing the series yields $2^{64} - 1$. □

4. George and Lucas are playing a game where they toss balls into three hoops. The smallest hoop is worth 17 points, the middle-sized hoop is worth 13 points, and the largest hoop is worth 9 points. George lost with only 84 points. What is the minimum number of balls he must have thrown?

   (A) 4
   (B) 5
   (C) 6
   (D) 7
   (E) 8



   SOLN Case 1: 5 balls in the 9 point hoop and 3 balls in the 13 point hoop.
   Case 2: 6 balls in the 9 point hoop, 1 ball in the 17 point hoop, and 1 ball in the 13 point hoop.
   For both of these cases, 8 balls were necessary to achieve the score of 84. □

5. Assume the following statements are true.
   - If it rains, we will bake a cake.
   - If it doesn't rain, we will play football.
   - If we play football, I will get muddy.
   - Mother will get angry if I get muddy.
   - I did not get muddy.

   Which is a valid conclusion?

   (A) Mother is not angry.
   [(B)] We baked a cake.
   (C) We did not bake a cake.
   (D) It did not rain.
   (E) We played football.

   [SOLN] If I didn't get muddy then we didn't play football, and it rained. □

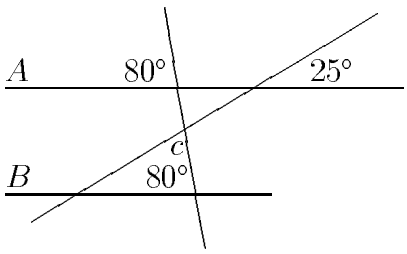
6. What is 50% more than twice the sum of 12 and 18?

   (A) 30
   (B) 45
   (C) 60
   [(D)] 90
   (E) 120

   [SOLN] Twice the sum of 12 and 18 is 60. 50% more than 60 is 90. □

7. If lines A and B are parallel ($A \parallel B$), then what is the measure of angle c?

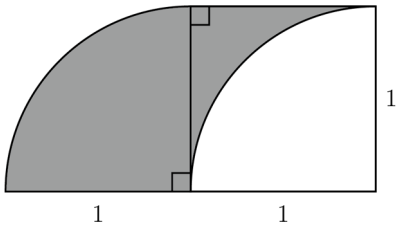
   (A) 65°
   (B) 70°
   [(C)] 75°
   (D) 80°
   (E) 90°



   [SOLN] $180 - 80 - 25 = 75$ □

8. The two arcs are quarter circles. What is the area of the shaded region?

   (A) π
   (B) $\frac{\pi}{4}$
   (C) $\frac{\pi}{4} + \frac{1}{2}$
   [(D)] 1
   (E) 4



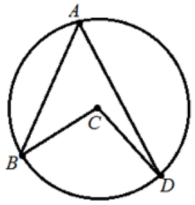
   [SOLN] The shaded quarter circle on the left just fills in the empty part of the square on the right. □

9. When a flock of sheep are driven through Sanpete Sally's land near Ephraim, she charges a toll of 10¢ per riderless animal (i.e., the sheep and dogs) and 50¢ for each rider and horse pair. One day Sally counted a total of 4168 legs (including riders, horses, dogs, and sheep) and 1044 heads. How much money did Sally collect?

   [(A)] $105.60
   (B) $123.80
   (C) $305.20
   (D) $518.40
   (E) $961.00

   [SOLN] legs: $2r + 4(h + s + d) = 4168$
   heads: $r + (h + s + d) = 1044$
   Solve head equation for $(h+s+d)$ and insert into the leg equation.
   $2r + 4(1044 - r) = 4168 \implies r = 4$
   4–legged animals $= 1044 - 4 = 1040$
   $s + d = 1040 - h = 1040 - 4 = 1036$
   Total toll $= 1036(\$0.10) + 4(\$0.50) =$ $105.60 □

10. Given that $m\angle BAD = x°$, find $m\angle BCD$. Point C is the center of the circle.



(A) $2x°$
(B) $x + 36°$
(C) $2x - 8°$
(D) $2x + 14°$
(E) $x + 40°$

SOLN Draw in radius $\overline{AC}$ and note that $\triangle ABC$ and $\triangle ACD$ are isosceles triangles with side length r. Using the sum theorem for a triangle in a plane, $m\angle BCD = 2x°$. □

11. Ruby has 10 white cubes and 17 red cubes, each 1 inch on each side. She arranges them to form a larger cube that is 3 inches on each side. What is the largest possible fraction of red surface area on the larger cube?

(A) $\frac{17}{27}$
(B) $\frac{7}{9}$
(C) 1
(D) $\frac{8}{9}$
(E) $\frac{2}{3}$

SOLN Of the smaller cubes, 8 are corner cubes (3 faces each), 12 non-corner edges (2 faces each), and 6 "side" cubes (1 face each). To maximize red surface area, the 8 corner cubes and 9 of the 12 non-corner edge cubes should be red, making $8 \cdot 3 + 9 \cdot 2 = 42$ of the possible 54 faces red. □

12. A rectangle has opposite vertices at $(1, -1)$, and $(3, 5)$. The other two vertices lie on the line $y = 2$. Find their coordinates.

(A) $(5, 2)$ and $(-1, 2)$
(B) $(2 - \sqrt{10}, 2)$ and $(2 + \sqrt{10}, 2)$
(C) $(2, 1)$ and $(2, 5)$
(D) $(4, 2)$ and $(0, 2)$
(E) $(3 - \sqrt{5}, 2)$ and $(3 + \sqrt{5}, 2)$

SOLN Perpendicular lines have negative reciprocal slopes: $\frac{5-2}{3-x} = \frac{1-x}{-(-1-2)}$. Solving this equation by the quadratic formula gives $x = 2 \pm \sqrt{10}$. □

13. What is the range of the function? $f(x) = \sqrt{8 \sin^3 x + 17}$

(A) $[0, \infty)$
(B) $[0, 5]$
(C) $[0, 23]$
(D) $[3, 5]$
(E) $[3, 23]$

SOLN The range of $\sin x$ is $[-1, 1]$; the range of that cubed is also $[-1, 1]$. Check left end: $\sqrt{-8 + 17} = 3$; right end: $\sqrt{8 + 17} = 5$. □

14. Suppose that $f(1 + x) = f(x)$ for all real x. If f is a polynomial and $f(4) = 5$, then what is $f(\frac{7}{2})$?

(A) -5
(B) 0
(C) 5
(D) $\frac{5}{2}$
(E) Not enough information

SOLN This function has a value of 5 at every integer x. It can't get infinitely big between integers and still be a polynomial. The only polynomials that are bounded as $x \to \pm\infty$ are of the type $f(x) = c$. □

15. The average of the ages of the mother, father, and three children is 21, while the average of the children is 11. How old is the father if he is 4 years older than the mother?

(A) 36

(B) 37

[(C)] 38

(D) 39

(E) 40

[SOLN] Let x be the age of the father. $x + (x-4) + 3 \cdot 11 = 5 \cdot 21$ □

16. A college math class has N teaching assistants. It takes the assistants 5 hours to grade homework assignments. One day, another teaching assistant joins them in grading, and all the assignments take only 4 hours to grade. Assuming everyone did the same amount of work, compute the number of hours it would take 1 teaching assistant to grade all the homework assignments.

[(A)] 20

(B) 22

(C) 24

(D) 26

(E) 28

[SOLN] Let W be the work it takes to grade all the assignments. We have $\frac{W}{N} = 5$, $\frac{W}{N+1} = 4$. $5N = 4N + 4 \implies N = 4$. If it takes 4 TAs 5 hours, then it takes 1 TA $4 \cdot 5$ hours. □

17. Suppose f is a linear function such that $(f \circ f \circ f)(x) = 27x + 26$. Find the y-intercept for the graph of f.

[(A)] $(0, 2)$

(B) $(26, 0)$

(C) $(0, 26)$

(D) $(0, 13)$

(E) $(0, 3)$

[SOLN] $f(x) = mx + b \implies (f \circ f)(x) = m(mx + b) + b = m^2x + bm + b \implies (f \circ f \circ f)(x) = m(m^2x + bm + b) + b = m^3x + bm^2 + bm + b$. Equating this to $27x + 26$ we see $m = 3$ so that $27x + 26 = 27x + 13b \Rightarrow b = 2$ □

18. For a closed rectangular box $L + W + H = 25\,\text{cm}$. The surface area of the box is $225\,\text{cm}^2$. Find the distance between opposite corners of the box; i.e., the largest distance between points on the box.

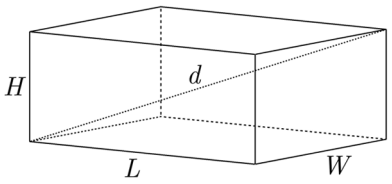
(A) 12.5 cm

(B) 15 cm

[(C)] 20 cm

(D) 25 cm

(E) 27.5 cm



[SOLN] $d = \sqrt{L^2 + W^2 + H^2}$. Since $(L + W + H)^2 = L^2 + W^2 + H^2 + 2LW + 2LH + 2WH = d^2 + 625$, we have $625 = d^2 + 225 \implies d = 20$. □

19. Consider three circles of radius 1, each tangent to the others. What is the radius of a fourth smaller circle in the middle which is tangent to each of them?

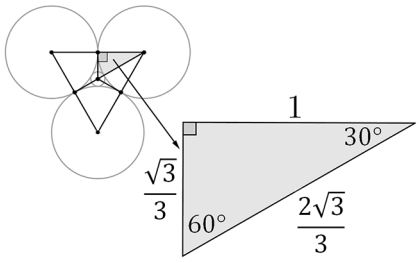
(A) $\frac{1}{4}$

(B) $\frac{\sqrt{3}-1}{2}$

(C) $\frac{\sqrt{3}-1}{4}$

(D) $\frac{\sqrt{3}}{3}$

(E) $\frac{2\sqrt{3}-3}{3}$



SOLN Drawing line segments as shown, we can create at 30-60-90 triangle with hypotenuse of length $2\sqrt{3}/3$. Since this hypotenuse is the distance between the center of the small circle and the center of a larger circle, the deisred result will be $2\sqrt{3}/3 - 1$. □

20. What is the area of a circle whose diameter is 2π cm?

(A) $\pi\,\text{cm}^2$

(B) $\pi\,\text{cm}^3$

(C) $\pi^2\,\text{cm}$

(D) $\pi^3\,\text{cm}^2$

(E) $4\pi\,\text{cm}^2$

SOLN $D = 2\pi\,\text{cm} \implies r = \pi\,\text{cm}$. $A = \pi r^2 = \pi(\pi\,\text{cm})^2$. Movement to replace 2π with τ: www.tauday.com □